Research Statement

Xavier Ramos Olivé

1 Overview

Since the 1950's, studying how local properties of a manifold, like curvature, affect global properties has drawn a lot of attention in Riemannian Geometry. Assuming that a manifold satisfies certain curvature bounds has consequences on topological invariants, as well as on analytic properties like the eigenvalues of the Laplacian or the solutions to the heat equation. For example, the Gauss-Bonnet formula on a closed surface S gives the expression for the Euler characteristic

$$2\pi\chi(S) = \int_S \sigma dv,$$

where σ is the sectional curvatue of S. Note that $\chi(S)$ is a topological invariant, a global property, while σ is a smooth function on S, a local property. Having a uniform lower bound for σ gives us a constraint on the possible values of $\chi(S)$. There is an extensive literature of results that focus on manifolds with a uniform lower bound on the Ricci curvature (an average of sectional curvatures in higher dimensions). However, many topological invariants can be expressed using L^p norms of the curvature, as is the case of the Euler characteristic above. This suggests that an integral bound on the curvature should be enough to estimate these topological invariants. Integral bounds on the curvature are much weaker assumptions than uniform pointwise bounds. They are also more suitable in the context of geometric flows like the Ricci flow. This motivates the study of manifolds with integral curvature assumptions, which is the main focus of my research.

In particular, I have worked on eigenvalue estimates for the Laplacian (see **Theorem 1.1**) and an estimate for the Neumann heat kernel (see **Theorem 1.2**) on manifolds with small integral Ricci curvature.

1.1 Background

To be more precise, consider a Riemannian manifold (M^n, g), i.e. a smooth n-dimensional manifold, that has a smoothly varying inner product $g_x(\cdot,\cdot)$ at the tangent space T_xM of every point $x \in M$. Let $\mathrm{Ric}_x(\cdot,\cdot)$ be the Ricci curvature tensor at x (a local property, that you can think of as a smooth and symmetric $n \times n$ matrix-valued function on M that depends on g). For $K \in \mathbb{R}$, we say that a manifold satisfies the pointwise lower bound $\mathrm{Ric} \geq (n-1)K$ if for any point $x \in M$ and for any vector $v_x \in T_xM$ we have that $\mathrm{Ric}_x(v_x, v_x) \geq (n-1)Kg_x(v_x, v_x)$. Since the Ricci curvature is symmetric, this is equivalent to saying that the lowest eigenvalue, $\rho(x)$, of Ric_x is uniformly bounded below by $(n-1)K$. Consider $\rho_K(x) := \max\{0, (n-1)K - \rho(x)\}$. This function gives the Ricci curvature below $(n-1)K$ at every point. Then, we can define

$$\bar{k}(p,K) = \left(\frac{1}{\mathrm{vol}(M)}\int_M \rho_K^p dv\right)^{\frac{1}{p}} = \left(\fint_M \rho_K^p dv\right)^{\frac{1}{p}},$$

$$\bar{k}(p,K,R) = \sup_{x\in M} R^2 \left(\fint_{B_R(x)} \rho_K^p dv\right)^{\frac{1}{p}}.$$

These quantities measure the total amount of Ricci curvature below $(n-1)K$ in an L^p sense. The first expression is a definition for compact manifolds, while the second one is also valid in the non-compact case. Notice that, in either case, $\bar{k} = 0$ if and only if $\mathrm{Ric} \geq (n-1)K$.

1.2 Eigenvalues of the Laplacian

A classical result from Lichnerowicz [Lic58] states that on a closed Riemannian manifold (M^n, g) satisfying $\mathrm{Ric} \geq (n-1)K$ for $K \geq 0$, the first nonzero eigenvalue $\lambda_1(M)$ of the Laplace-Beltrami operator satisfies $\lambda_1(M) \geq nK$. In the case $\mathrm{Ric} \geq 0$, one needs control on the diameter D to get a meaningful estimate. Li and Yau [LY80] proved that, in that case, $\lambda_1(M) \geq \frac{\pi^2}{4D^2}$. Zhong and Yang [ZY84] improved this result showing that

$$\lambda_1(M) \geq \frac{\pi^2}{D^2}.$$

This estimate is sharp, since the lower bound is realized by S^1. In fact, rigidity was proven in [HW07]: if a manifold with non-negative Ricci curvature realizes the lower bound, then it is isometric to S^1 with radius $\frac{D}{\pi}$.

In a joint work with Shoo Seto, Guofang Wei and Qi S. Zhang, we proved an analogous result for manifolds with small integral Ricci curvature. The result is sharp in the sense that it recovers the Zhong-Yang estimate in the limit where $\mathrm{Ric} \geq 0$.

Theorem 1.1 ([RSWZ]). *Let (M^n, g) be a closed Riemannian manifold with diameter $\leq D$ and $\lambda_1(M)$ be the first nonzero eigenvalue of the Laplacian. For any $\alpha \in (0,1)$, $p > \frac{n}{2}$, there exists $\epsilon(n, p, \alpha, D) > 0$ such that if $\bar{k}(p, 0) \leq \epsilon$, then*

$$\lambda_1(M) \geq \alpha \frac{\pi^2}{D^2}.$$

In general, the smallness of $\bar{k}(p, 0)$ is a necessary condition, as explained in [DWZ18], because assuming only that $\bar{k}(p, 0)$ is bounded imposes almost no restriction on the manifold. In this particular case, we can see that it is necessary with the example of the dumbbell of Calabi: consider a symmetric dumbbell $D_\epsilon \subseteq \mathbb{R}^3$, formed by two equal spheres joined by a thin cylinder of radius ϵ and length l, with joints smoothed out in two equal neck regions. Assume without loss of generality that the area of D_ϵ is less than 1. As explained by Cheeger in [Che70], the first nonzero eigenvalue satisfies $\lambda_1(D_\epsilon) \leq C\epsilon^2$ for some constant C, thus there's no possible uniform lower bound on $\lambda_1(D_\epsilon)$, since we can consider sequences of dumbbells with $\epsilon \to 0$.

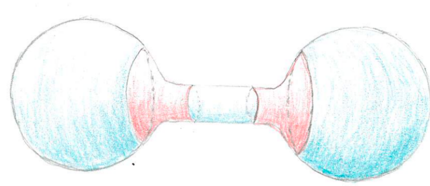

Fig. 1. Dumbbell of Calabi. The neck regions (pink) have $\bar{k}(1, 0) \approx 4\pi$.

Although in this example $\bar{k}(p, 0)$ could be uniformly bounded, it can not be made small. The reason for that lies in the Gauss-Bonnet formula: since the dumbbell is homeomorphic to the sphere, the integral of the curvature over D_ϵ must be 4π. However, the integral over the two spheres will be close to 8π, and over the cylinder will be 0. Hence, over the two neck regions we will have negative curvature, that will be close to -4π in total, to be able to cancel the curvature coming from the spheres. Thus, the L^1 norm of ρ_0 is close to 4π. Therefore, using Hölder's inequality, we obtain a lower bound on $\bar{k}(p, 0)$ for any $p > 1 = \frac{n}{2}$.

1.3 Heat equation

Another example of a result that follows from assuming a Ricci curvature lower bound is the Li-Yau gradient estimate for positive solutions to the heat equation. In the celebrated paper [LY86], Li and Yau proved that if $u > 0$ is a solution to the heat equation on a convex domain Ω (i.e. the second fundamental form of the boundary satisfies $II \geq 0$) of a compact manifold M satisfying $\mathrm{Ric} \geq (n-1)K$, with Neumann boundary conditions, then for all $\alpha > 1$

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq C_1 + C_2 \frac{1}{t},$$

where $C_i = C_i(n, \alpha, K)$. Later, Wang [Wan97] proved that the same result holds for non-convex domains, assuming that $II \geq -H$ for some $H \geq 0$, and adding the necessary *interior rolling R-ball* condition: that there exists $R > 0$ small enough so that for every point p in the boundary, there exists a ball $B(q, R) \subseteq \Omega$ that intersects the boundary only at p. Here, the smallness of R depends on the sectional curvature near the boundary.

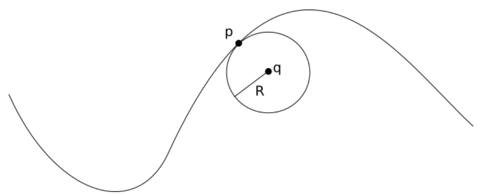

Fig. 2. Interior rolling R-ball condition.

In a similar spirit, I proved the following result for the integral curvature condition case.

Theorem 1.2 ([Ram19]). *Given $H, n > 0$, $p > \frac{n}{2}$, and $R > 0$ small enough, there exists $\epsilon(n,p) > 0$ such that if $\bar{k}(p, 0, D) \leq \epsilon$, $II \geq -H$, and the interior rolling $R-$ball condition holds in Ω, then any $u > 0$ solving the heat equation with Neumann boundary conditions satisfies the Li-Yau type gradient estimate*

$$\alpha \underline{J} \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq C_1 + \frac{C_2}{\underline{J}} \frac{1}{t}$$

for any $0 < \alpha < \frac{1}{(1+H)^2}$, $C_i = C_i(n, \alpha, H, R)$, and $\underline{J}(t) = C(\alpha, n) e^{-C_3 t}$, with $C_3 = C_3(\alpha, n, p, D, K, R)$.

As a corollary, following the same method as [LY86], one can obtain the following Harnack inequality.

Corollary 1.1 (Harnack inequality). *If M, Ω and u are as above, with Ω path connected, $x, y \in \Omega$, $t_2 \geq t_1 > 0$, we have*

$$u(x, t_1) \leq u(y, t_2) \left(\frac{t_2}{t_1}\right)^{\frac{C_2}{\underline{J}(t_2)}} \exp\left(C_1(t_2 - t_1) + \frac{\inf_{\gamma \in \Gamma} \int_0^1 |\dot{\gamma}|^2 ds}{4\alpha \underline{J}(t_2)(t_2 - t_1)}\right),$$

where Γ is the set of smooth curves $\gamma : [0,1] \to \Omega$ with $\gamma(0) = y$ and $\gamma(1) = x$.

2 The method: gradient estimates

Both Theorem 1.1 and Corollary 1.1 share something in common: the key to their proofs is a gradient estimate proved using a maximum principle. Many important results in Geometric Analysis rely on this technique. Very broadly speaking, the general strategy to prove one of these estimates when $\mathrm{Ric} \geq (n-1)K$ is the following:

1. Define a suitable function $F(x) \approx |\nabla \ln(u(x))|^2$.

2. Let m be the maximum of F in M.
3. Show that m is an interior point, hence $\nabla F(m) = 0$ and $\Delta F(m) \leq 0$.
4. Use Bochner's formula: $\frac{1}{2}\Delta|\nabla u|^2 = g(\nabla \Delta u, \nabla u) + |\text{Hess } u|^2 + \text{Ric}(\nabla u, \nabla u)$.
5. Use that $\text{Ric} \geq (n-1)K$ to eliminate the curvature terms.
6. Conclude the result, after several elementary inequalities.

In [ZZ17] and [ZZ18], Zhang and Zhu introduced an auxiliary function J in the definition of F to be able to prove, among other results, several Li-Yau gradient estimates under integral curvature conditions. The idea is that in step 5, instead of using that $\text{Ric}(\nabla u, \nabla u) \geq (n-1)K|\nabla u|^2$, we can use that $\text{Ric}(\nabla u, \nabla u) \geq -\rho_0|\nabla u|^2$, and then we can group this term together with all the terms involving derivatives of J and set them equal to a suitable expression. This way, we define J using a differential equation, and the curvature assumption gets absorbed in the definition of J. The next step is to control the behavior of J to be able to derive a useful estimate. We are able to accomplish this using the volume comparison of [PW97], the Sobolev inequalities of [DWZ18], [Gal88] and [PS98], Gaussian estimates for the heat kernel as in [CKO15], and Moser's iteration as in [WY09] and [HL11], among other techniques.

2.1 Zhong-Yang estimate

For example, in the case of Theorem 1.1, we define J to be a solution of:

$$\Delta J - \tau \frac{|\nabla J|^2}{J} - 2\rho_0 J = -\sigma J, \tag{1}$$

for $\tau > 1$, $\sigma \geq 0$. Note that in the limit $\text{Ric} \geq 0$, we can choose $\sigma = 0$ and $J \equiv 1$. Then we prove that there exists solutions to the equation above for which we have good control on J and σ, namely:

Lemma 2.1 ([RSWZ]). *For any $\delta > 0$, there exists $\epsilon(n, p, D, \tau) > 0$, $\sigma \geq 0$ and a solution J such that if $\bar{k}(p, 0) \leq \epsilon$, then*

$$0 \leq \sigma \leq 4\epsilon,$$

and

$$|J - 1| \leq \delta.$$

The idea of this lemma is that we can choose J and σ to be a perturbation of the solution in the case $\text{Ric} \geq 0$ mentioned above. Equation (1) can be turned into a linear equation using $J = w^{-\frac{1}{\tau-1}}$. Then we choose a specific solution to the equation for w, and through different techniques, we estimate σ and w, which in turn gives us estimates on J.

With this estimates on J, we can formulate the key gradient estimate used to prove Theorem 1.1. Let u be a nontrivial solution of $-\Delta u = \lambda_1 u$ on a closed Riemannian manifold M. Suppose without loss of generality that $\sup u = k + 1$ and $\inf u = k - 1$, for $0 \leq k < 1$. Set $v := u - k$, so that $-\Delta v = \lambda_1(v + k)$.

Theorem 2.1 (Gradient estimate, [RSWZ]). *For a fixed $\delta > 0$, $\tau = \tau(\delta)$, suppose $\bar{k}(p, 0) \leq \epsilon(n, p, D, \delta)$ as in the Lemma above. Then v satisfies*

$$J|\nabla v|^2 \leq \tilde{\lambda}(1 - v^2) + 2k\lambda_1\psi(v),$$

where $\tilde{\lambda} := C_1\lambda_1 + C_2$ with $C_1 = C_1(\delta)$, $C_2 = C_2(\delta, \sup\psi, \sigma)$, and ψ is explicitly defined through an ODE (inspired by the approach of [Li12]*).*

Then, following a standard technique (see for instance [SY94]), integrating along a geodesic, we get the estimate in Theorem 1.1.

2.2 Li-Yau estimate

In the case of Theorem 1.2, we define $J(x,t)$ to be the unique smooth solution of:

$$\begin{cases} \Delta J - \partial_t J - \tau \frac{|\nabla J|^2}{J} - 2\rho_0 J = 0 & \text{in } \Omega \times (0,\infty) \\ \partial_\nu J = 0 & \text{on } \partial\Omega \times (0,\infty) \\ J = 1 & \text{on } \Omega \times \{0\} \end{cases} \tag{2}$$

for $\tau > 1$. Using Neumann boundary conditions for J, we can use essentially the same technique as [Wan97] to show that the maximum m must be an interior point (step 3 of the method mentioned above). Then we show that we can control J, namely

Lemma 2.2 ([Ram19]). *If J solves equation* (2)*, we have*

$$0 < \underline{J} \leq J \leq 1,$$

where $\underline{J} = \underline{J}(t)$ is given by

$$\underline{J}(t) := 2^{-\frac{1}{c-1}} e^{-\frac{\widetilde{C_3}}{c-1}t},$$

and $\widetilde{C_3} = C_3(c,n,p)\left[\frac{K}{D^{2-\frac{n}{p}} R^{\frac{n}{p}}} + \frac{K^{\frac{2p}{2p-n}}}{D^{\frac{4p-6n}{2p-n}} R^{\frac{4n}{2p-n}}}\right] > 0.$

As in the Zhong-Yang estimate, the transformation $J = w^{-\frac{1}{\tau-1}}$ turns equation (2) into a linear equation for w, that we can estimate using the Gaussian estimates for the heat kernel from [CKO15], and this gives us the estimates for J. To accomplish this we need to show that the volume doubling property from [PW97] is satisfied up to the boundary. This could fail if the boundary was very irregular, but the interior rolling $R-$ball condition ensures that this is not the case (which could be useful in other contexts). More precisely, we have the following lemma.

Lemma 2.3 ([Ram19]). *Given $D, \gamma > 0$, suppose that a Riemannian manifold (M^n, g) satisfies the volume doubling property*

$$V(x,s) \leq C\left(\frac{s}{r}\right)^\gamma V(x,r) \tag{3}$$

for $r \leq s \leq D$ and $x \in M$, where $V(x,r) = \text{vol}(B(x,r))$. Then a compact domain $\Omega \subseteq M$, with diam$(\Omega) \leq D$, whose boundary satisfies the interior rolling $R-$ball condition, satisfies:

$$V_\Omega(x,s) \leq \widetilde{C}\left(\frac{s}{r}\right)^\gamma V_\Omega(x,r) \tag{4}$$

for $0 < r \leq s$, $x \in \Omega$, $\widetilde{C} = \max\{3^\gamma C, \left(\frac{2D}{R}\right)^\gamma C\}$, where $V_\Omega(x,r) = \text{vol}(B(x,r) \cap \Omega)$.

3 Future directions

3.1 Neumann eigenvalues on star-shaped domains

Chen and Li [CL97] showed that if a Riemannian manifold with boundary is geodesically star-shaped with respect to $p \in M$, with Ric $\geq -(n-1)K$ for $K \geq 0$, the first nonzero Neumann eigenvalue η_1 has lower bound given by

$$\eta_1 \geq \frac{R^n}{R_0^{n+2}} \exp(-C_1(1 + R_0\sqrt{K})),$$

where R and R_0 are the radii of the geodesic balls centered at p of largest radius contained in M and of smallest radius containing M, respectively. Their proof relies on Neumann heat kernel estimates, that follow from the Li-Yau gradient estimate of [LY86], that Theorem 1.2 generalizes to the integral Ricci curvature case. For this reason, I want to study if a similar bound applies under integral curvature conditions.

3.2 Fundamental gap

Another kind of results that I plan to study are estimates on the fundamental gap, i.e. the difference between the first two eigenvalues of the Laplacian or the Schrödinger operators (either in closed manifolds or on domains with Dirichlett or Neumann boundary conditions). In fact, the estimate of Chen and Li mentioned above could be regarded as a fundamental gap estimate for star-shaped domains, since under Neumann boundary conditions we have that $\eta_0 = 0$ is the first eigenvalue. This problem has been studied very extensively. Lower bounds in convex domains of Euclidean space were shown in [SWYY85], [YZ86], and a sharp lower bound was proven in [AC11]. In convex domains of a sphere, a sharp lower bound was given in [SWW16]. For manifolds with $\mathrm{Ric} \geq -(n-1)K$ for $K \geq 0$, sharp lower bounds in the Neumann case were proven by several authors. In closed manifolds, for $K = 0$, this is the result of [ZY84] mentioned in §1, of which Theorem 1.1 provides a generalization to integral curvature. In view of that, a natural question is to study sharp lower bounds of the fundamental gap for convex domains in the Neumann and Dirichlett cases under integral curvature conditions.

3.3 Integral Bakry-Émery case

Consider a smooth function f. Many of the results mentioned here have equivalent formulations for the drift Laplacian $\Delta_f = \Delta + \nabla f \nabla$ under lower bounds on the Bakry-Émery Ricci tensor $\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess} f$, as in [AN12] or [CLR15]. Since some of the main analytic tools needed are known to be true under integral Bakry-Émery conditions, as in [Wu16] or [WW], it is natural to study eigenvalue estimates in this setting as well. In particular, I would like to study a sharp lower bound like the one in Theorem 1.1 for the first eigenvalue of Δ_f.

3.4 Kähler case

Another natural direction would be the Kähler case. On Kähler manifolds, under $\mathrm{Ric} \geq -(n-1)K$, sometimes the eigenvalue estimates can be improved, as in [GKY13]. The Ricci curvature in this case can be written as $R_{i\bar{j}} = \partial_i \partial_{\bar{j}} u + g_{i\bar{j}}$ for some Kähler-Ricci potential u. Assuming appropriate integral conditions on u, I want to prove some eigenvalue estimates without assuming a pointwise lower bound in the curvature.

3.5 Higher order eigenvalue estimates

Buser [Bus82] showed that under a pointwise lower bound on the Ricci curvature, λ_1 could be bounded above by an expression depending on the Cheeger constant $h(M)$. Recently, Benson [Ben15] generalized this technique to give upper bounds on all the eigenvalues $\lambda_k(M)$ in terms of the Cheeger constant, using Sturm-Liouville theory. One of the key ingredients of this work is the Heintze-Karcher inequality, which was studied in the integral curvature setting in [PS98]. I plan to study these upper bounds under integral Ricci curvature conditions.

References

[AC11] Ben Andrews and Julie Clutterbuck, *Proof of the fundamental gap conjecture*, J. Amer. Math. Soc. **24** (2011), no. 3, 899–916, DOI 10.1090/S0894-0347-2011-00699-1. MR2784332

[AN12] Ben Andrews and Lei Ni, *Eigenvalue comparison on Bakry-Emery manifolds*, Comm. Partial Differential Equations **37** (2012), no. 11, 2081–2092, DOI 10.1080/03605302.2012.668602. MR3005536

[Ben15] Brian Benson, *Sturm-Liouville estimates for the spectrum and Cheeger constant*, Int. Math. Res. Not. IMRN **16** (2015), 7510–7551, DOI 10.1093/imrn/rnu175. MR3428972

[Bus82] Peter Buser, *A note on the isoperimetric constant*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 2, 213–230. MR683635

[CLR15] Nelia Charalambous, Zhiqin Lu, and Julie Rowlett, *Eigenvalue estimates on Bakry-Émery manifolds*, Elliptic and parabolic equations, Springer Proc. Math. Stat., vol. 119, Springer, Cham, 2015, pp. 45–61, DOI 10.1007/978-3-319-12547-3$_2$. MR3375166

[Che70] Jeff Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in analysis (Papers dedicated to Salomon Bochner, 1969), Princeton Univ. Press, Princeton, N. J., 1970, pp. 195–199. MR0402831

[CL97] Roger Chen and Peter Li, *On Poincaré type inequalities*, Trans. Amer. Math. Soc. **349** (1997), no. 4, 1561–1585, DOI 10.1090/S0002-9947-97-01813-8. MR1401517

[CKO15] Mourad Choulli, Laurent Kayser, and El Maati Ouhabaz, *Observations on Gaussian upper bounds for Neumann heat kernels*, Bull. Aust. Math. Soc. **92** (2015), no. 3, 429–439, DOI 10.1017/S0004972715000611. MR3415619

[DWZ18] Xianzhe Dai, Guofang Wei, and Zhenlei Zhang, *Local Sobolev constant estimate for integral Ricci curvature bounds*, Adv. Math. **325** (2018), 1–33, DOI 10.1016/j.aim.2017.11.024. MR3742584

[Gal88] Sylvestre Gallot, *Isoperimetric inequalities based on integral norms of Ricci curvature*, Astérisque **157-158** (1988), 191–216. Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987). MR976219

[GKY13] Vincent Guedj, Boris Kolev, and Nader Yeganefar, *A Lichnerowicz estimate for the first eigenvalue of convex domains in Kähler manifolds*, Anal. PDE **6** (2013), no. 5, 1001–1012, DOI 10.2140/apde.2013.6.1001. MR3125547

[HL11] Qing Han and Fanghua Lin, *Elliptic Partial Differential Equations*, Courant Lecture Notes, vol. 1, American Mathematical Society, 2011. MR1669352

[HW07] Fengbo Hang and Xiaodong Wang, *A remark on Zhong-Yang's eigenvalue estimate*, Int. Math. Res. Not. IMRN **18** (2007), Art. ID rnm064, 9, DOI 10.1093/imrn/rnm064. MR2358887

[Li12] Peter Li, *Geometric analysis*, Cambridge Studies in Advanced Mathematics, vol. 134, Cambridge University Press, Cambridge, 2012. MR2962229

[LY80] Peter Li and Shing-Tung Yau, *Estimates of eigenvalues of a compact Riemannian manifold*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 205–239. MR573435

[LY86] ———, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), no. 3-4, 153–201, DOI 10.1007/BF02399203. MR834612

[Lic58] André Lichnerowicz, *Géométrie des groupes de transformations*, Travaux et Recherches Mathématiques, III. Dunod, Paris, 1958 (French). MR0124009

[PS98] Peter Petersen and Chadwick Sprouse, *Integral curvature bounds, distance estimates and applications*, J. Differential Geom. **50** (1998), no. 2, 269–298. MR1684981

[PW97] P. Petersen and G. Wei, *Relative volume comparison with integral curvature bounds*, Geom. Funct. Anal. **7** (1997), no. 6, 1031–1045, DOI 10.1007/s000390050036. MR1487753

[Ram19] Xavier Ramos Olivé, *Neumann Li-Yau gradient estimate under integral Ricci curvature bounds*, Proc. Amer. Math. Soc. **147** (2019), no. 1, 411–426, DOI 10.1090/proc/14213.

[RSWZ] Xavier Ramos Olivé, Shoo Seto, Guofang Wei, and Qi S. Zhang, *Zhong-Yang type eigenvalue estimate with integral curvature condition*, preprint.

[SY94] R. Schoen and S.-T. Yau, *Lectures on differential geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, I, International Press, Cambridge, MA, 1994. Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu; Translated from the Chinese by Ding and S. Y. Cheng; With a preface translated from the Chinese by Kaising Tso. MR1333601

[SWW16] Shoo Seto, Lili Wang, and Guofang Wei, *Sharp Fundamental Gap Estimate on Convex Domains of Sphere* (2016), available at `arXiv:1606.01212`.

[SW17] Shoo Seto and Guofang Wei, *First eigenvalue of the p-Laplacian under integral curvature condition*, Nonlinear Anal. **163** (2017), 60–70.

[SWYY85] I. M. Singer, Bun Wong, Shing-Tung Yau, and Stephen S.-T. Yau, *An estimate of the gap of the first two eigenvalues in the Schrödinger operator*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), no. 2, 319–333. MR829055

[Wan97] Jiaping Wang, *Global heat kernel estimates*, Pacific J. Math. **178** (1997), no. 2, 377–398, DOI 10.2140/pjm.1997.178.377. MR1447421

[WW] Lili Wang and Guofang Wei, *Local Sobolev constant estimate for integral Barky-Emery Ricci curvature*, available at `http://mail.math.ucsb.edu/~wei/paper/Wang-Wei.pdf`.

[Wu16] Jia-Yong Wu, *Comparison Geometry for integral Bakry-Émery Ricci tensor bounds* (2016), available at `arXiv:1610.03926`.

[WY09] Guofang Wei and Rugang Ye, *A Neumann type maximum principle for the Laplace operator on compact Riemannian manifolds*, J. Geom. Anal. **19** (2009), no. 3, 719–736, DOI 10.1007/s12220-009-9080-0. MR2496575

[YZ86] Qi Huang Yu and Jia Qing Zhong, *Lower bounds of the gap between the first and second eigenvalues of the Schrödinger operator*, Trans. Amer. Math. Soc. **294** (1986), no. 1, 341–349, DOI 10.2307/2000135. MR819952

[ZZ17] Qi S. Zhang and Meng Zhu, *Li-Yau gradient bound for collapsing manifolds under integral curvature condition*, Proc. Amer. Math. Soc. **145** (2017), no. 7, 3117–3126, DOI 10.1090/proc/13418. MR3637958

[ZZ18] ———, *Li-Yau gradient bounds on compact manifolds under nearly optimal curvature conditions*, J. Funct. Anal. **275** (2018), no. 2, 478–515, DOI 10.1016/j.jfa.2018.02.001. MR3802491

[ZY84] Jia Qing Zhong and Hong Cang Yang, *On the estimate of the first eigenvalue of a compact Riemannian manifold*, Sci. Sinica Ser. A **27** (1984), no. 12, 1265–1273. MR794292