Research Statement

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1 Overview

My work is on Geometric Analysis. I study partial differential equations on manifolds to understand how local properties, like curvature, affect global properties, like the eigenvalues of the Laplacian.

There is an extensive literature of results that focus on manifolds with a pointwise lower bound on the Ricci curvature. My research, by contrast, focuses on integral curvature assumptions. These are much weaker assumptions than pointwise bounds, and they are more stable under perturbations of the geometry.

In particular, I have worked on eigenvalue estimates for the Laplacian (see **Theorem 3.1**, [RSWZ20] and [RRWW23]) and estimates for linear and non-linear heat equations (see [Ram19], [PRR23] and [RS23]) on manifolds with integral curvature conditions. I have also worked with undergraduate students on related research projects (see **Theorem 3.2** and [ARS21]). I am also interested in Ricci curvature notions defined on discrete spaces, like graphs, and their applications to Data Science. In fact, I will be coorganizing the *AMS Special Session on Ricci Curvatures of Graphs and Applications to Data Science (a Mathematics Research Communities session)* at the 2024 Joint Mathematics Meetings in San Francisco

2 Integral Curvature Assumptions vs Pointwise Bounds

To be more precise about the curvature conditions I study, consider a Riemannian manifold (M^n, g). Let $\mathrm{Ric}(x)$ be the lowest eigenvalue of the Ricci curvature tensor at $x \in M$. For $K \in \mathbb{R}$, we can compare the curvature of M with the one of a model space with constant curvature value $(n-1)K$ (a sphere if $K > 0$, Euclidean space if $K = 0$, or Hyperbolic space if $K < 0$). We say that a manifold satisfies a pointwise lower bound on the Ricci curvaute if $\mathrm{Ric} \geq (n-1)K$. Consider $\rho_K(x) := \max\{0, (n-1)K - \mathrm{Ric}(x)\}$, the function that gives the Ricci curvature below the threshold $(n-1)K$ at every point. Then, we can define

$$\bar{k}(p, K) = \left(\frac{1}{\mathrm{vol}(M)} \int_M \rho_K^p dv \right)^{\frac{1}{p}}.$$

This quantity measures the total amount of Ricci curvature below $(n-1)K$ in an L^p sense. Notice that $\bar{k} = 0$ if and only if $\mathrm{Ric} \geq (n-1)K$.

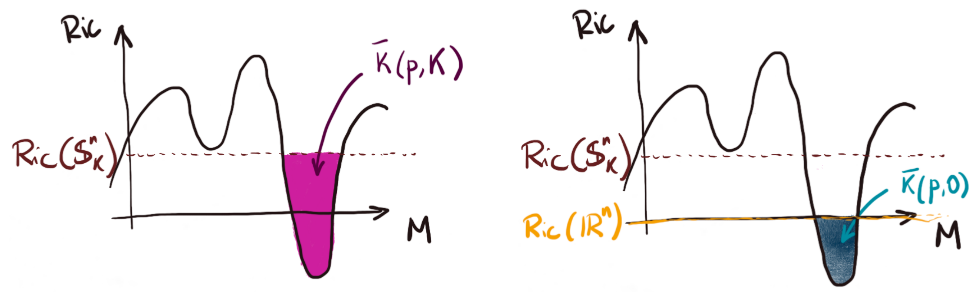

Figure 1: The value of $\mathrm{Ric}(x)$ is plotted and compared with the values of the Ricci curvature of a sphere $\mathrm{Ric}(\mathbb{S}^n_K) = (n-1)K$ and of Euclidean space $\mathrm{Ric}(\mathbb{R}^n) = 0$. The integral curvature $\bar{k}(p, K)$ is a way of measuring how much the inequality $\mathrm{Ric} \geq (n-1)K$ is violated.

3 Eigenvalues on closed manifolds

In this section I present the main results in two of my articles. A classical result from Lichnerowicz [Lic58] states that on a closed Riemannian manifold (M^n, g) satisfying $\mathrm{Ric} \geq (n-1)K$ for $K > 0$, the first nonzero eigenvalue $\lambda_1(M)$ of the Laplace-Beltrami operator satisfies $\lambda_1(M) \geq nK$. In the case $\mathrm{Ric} \geq 0$, one needs control on the diameter D to get a meaningful estimate. Li and Yau [LY80] proved that, in that case, $\lambda_1(M) \geq \frac{\pi^2}{2D^2}$. Zhong and Yang [ZY84] improved this result, proving the sharp estimate

$$\lambda_1(M) \geq \frac{\pi^2}{D^2}.$$

In the joint work [RSWZ20], we proved an analogous result under integral curvature assumptions. The result is sharp in the sense that it recovers the Zhong-Yang estimate in the limit $\bar{k}(p, 0) = 0$, i.e. $\mathrm{Ric} \geq 0$.

Theorem 3.1 ([RSWZ20]). *Let (M^n, g) be a closed Riemannian manifold with $\mathrm{diam}(M) \leq D$ and $\lambda_1(M)$ be the first nonzero eigenvalue of the Laplacian. For any $\alpha \in (0, 1)$, $p > \frac{n}{2}$, there exists $\epsilon(n, p, \alpha, D) > 0$ such that if $\bar{k}(p, 0) \leq \epsilon$, then*

$$\lambda_1(M) \geq \alpha \frac{\pi^2}{D^2}.$$

This theorem tells us that we can get as close as we want to the optimal estimate $\frac{\pi^2}{D^2}$ as long as the amount of negative Ricci curvature is small enough in an L^p sense.

It is well known to the experts that the smallness of $\bar{k}(p, 0)$ is a necessary condition to obtain a lower bound on λ_1, and cannot be replaced by a bound on $\bar{k}(p, 0)$ (see [Gal88]). This can be shown by constructing a sequence of manifolds for which $\bar{k}(p, 0)$ is uniformly bounded, and whose first nonzero eigenvalue approaches 0 along the sequence. However, the construction in [Gal88] might be hard to follow for people foreign to the subject. Because of this, I engaged two undergraduate students in the construction of a more elementary example, based on the dumbbell shaped surfaces of Calabi (see [Che70]). We wrote an article with this construction, which was published in [ARS21], and they presented these findings at *2021 MAA MathFest*. They are both currently pursuing their PhD in Mathematics.

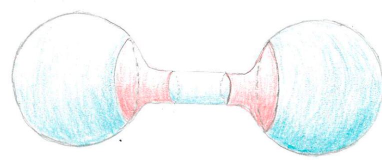

Figure 2: Dumbbell of Calabi. The surface has $\bar{k}(p, 0) \geq \bar{k}(1, 0) \approx 4\pi > 2\pi$, regardless of the sizes of the components. Thus, the integral curvature is not small enough for Theorem 3.1 to hold.

Theorem 3.2 ([ARS21]). *For any $K > 0$, there exist constants $p > 1$, $C > 0$, and $D > 0$, and a sequence of $\mathcal{C}^\infty$-surfaces $\{(\Sigma_i, g_i)\}_{i \in \mathbb{N}}$, such that $\mathrm{diam}(\Sigma_i) \leq D$, $\bar{k}(p, K)_{\Sigma_i} \leq C$, and*

$$\lim_{i \to \infty} \lambda_1(\Sigma_i) = 0.$$

4 Other research and new directions

In addition of studying the first non-zero eigenvalue λ_1, I have also worked on estimates for the Dirichlet fundamental gap $\Gamma = \lambda_2 - \lambda_1$ on non-convex domains (see [RRWW23]), for the Neumann heat kernel (see [Ram19]), for non-linear parabolic equations (see [RS23]), and for Sobolev extension operators (see [PRR23]).

4.1 Yang's estimate for integral curvature

Developing on the Zhong-Yang estimate, Yang proved that when $K < 0$, if $\mathrm{Ric} \geq (n-1)K$ and the diameter is bounded by D, then

$$\lambda_1 \geq \frac{\pi^2}{D^2} e^{-c_n \sqrt{|K|} D},$$

where c_n only depends on the dimension of the manifold n. This result has not been generalized yet to the setting of integral Ricci curvature conditions. One expects the proof to be similar to the proof of our Theorem 3.1. However, when we generalized Zhong-Yang's estimate, we used an alternative proof of the result due to Peter Li, which utilizes an ODE technique. As far as I know, no such proof exists of Yang's estimate, which makes adapting our theorem challenging.

I want to eventually generalize Yang's theorem to the setting of smooth weighted manifolds under integral assumptions on their Bakry-Émery Ricci curvature, a kind of spaces I worked on in my PhD thesis as well as in [RS23]. Weighted manifolds are Riemannian manifolds where one uses a general volume form which is absolutely continuous with respect to the Riemannian volume. They are widely studied due to their connection with gradient Ricci solitons, which play a central role in the study of Ricci flow.

4.2 Ricci curvature in discrete spaces

In recent years, there has been a growing interest for studying the Ricci curvature on more general metric measure spaces (not necessarily smooth manifolds), and in particular on graphs. Several classical results from Geometric Analysis can be translated to this setting. These new notions of Ricci curvature are useful in applications to data science and artificial intelligence. My current area of expertise is relevant here, since some of the problems that people are interested in include extremal problems on graphs satisfying curvature restrictions, or spectral applications to clustering and community detection algorithms. In 2023 I participated in the AMS Mathematics Research Community based on these topics.

One of the theorems that we hope to recover in this new setting is the rigidity result of Obata. On manifolds, we know from Lichnerowicz that if $\mathrm{Ric} \geq (n-1)K > 0$, then $\lambda_1 \geq nK$. Moreover, Obata proved that if $\lambda_1 = nK$, then the manifold is isometric to a sphere. If we consider the eigenvalues of the graph Laplacian, a similar result can be proven on graphs when considering the Lin-Lu-Yau Ricci curvature. We know that if $\mathrm{Ric} \geq K$ then $\lambda_1 \geq K$. The question is, for which graphs do we have that $\mathrm{Ric} \geq K$ and $\lambda_1 = K$?

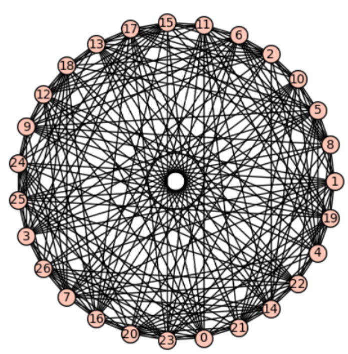

Figure 3: The Schläfli graph has $\lambda_1 = 3/4$ and every edge in it has Lin-Lu-Yau Ricci curvature value $3/4$.

Working with Linyuan Lu, we initially speculated that only complete graphs or their cartesian products could achieve the lower bound $\lambda_1 = K$. However, we disproved our conjecture when we discovered that the Schläfli graph also meets these conditions. This finding leads us to believe that a broader range of regular graphs exhibit this property. Investigating which graphs satisfy the optimal value of λ_1 presents an engaging opportunity for undergraduate student research. This project offers a balanced blend of theoretical mathematics and computational elements, requiring no more than a foundation in linear algebra. Its open-ended and exploratory nature is accompanied by a significant underlying open problem, making it interesting to experts in the field. This project can be adapted to other versions of Ricci curvature on graphs, and I look forward to guiding students to explore these problems in the near future.

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