PROBLEMS SECTION 2
MATH 10A, Summer Session E, UC Riverside 2018

Section 2.1

Problem 42. *Draw three level sets for the function* $f(x, y) = x + 2y + 1$ *at the levels* $C = -1, 0, 1$.

Problem 43. *Draw three level sets for the function* $g(x, y) = x^2 + (y - 1)^2 + 1$ *at the levels* $C = 1, 2, 5$. *What is the level set of value* $C = 0$*?*

Problem 44. *Draw three level sets for the function* $f(x, y) = x^2 + 4y^2$ *at the levels* $C = 0, 1, 4$.

Problem 45. *Draw three level sets for the function* $f(x, y) = x^2 - y^2$ *at the levels* $C = 0, 1, 4$.

Problem 46. *Draw the level curve of the function* $f(x, y) = x^2 + 2y^2$ *that contains the point* $(2, 1)$. *Draw the line tangent to this level curve at that point.*

Problem 47. *Draw the level curves of the function* $f(x, y) = 1 - |x| - |y|$. **Hint:** *consider the function in each of the* 4 *quadrants separately.*

Problem 48. *Describe (or sketch) the set of points in* $\mathbb{R}^2$ *satisfying the equations:*

(1) $y = 2x + 1$
(2) $x^2 + 4y^2 = 1$

Describe (or sketch) the set of point in $\mathbb{R}^3$ *satisfying the equations:*

(1) $y = 2x + 1$
(2) $x^2 + 4y^2 = 1$

Problem 49. *Describe (or sketch) the sets of points in* $\mathbb{R}^3$ *that satisfy the equations:*

(1) $3x - z = 4$
(2) $x^2 + y^2 + z^2 = 4$
(3) $4x^2 + 9y^2 + z^2 = 36$
(4) $(x - 1)^2 + (y + 1)^2 + z^2 = 9$
(5) $x^2 + y^2 = z^2$
(6) $x^2 + y^2 = z^2 + 4$
(7) $x^2 + y^2 = z^2 - 4$
(8) $x^2 + y^2 = z$

Hint: *in case of doubt, intersect the surfaces with the planes* $z = C$ *for different values of* C*, and with the planes* $x = 0$ *or* $y = 0$ *(specially recommended in the last four). You can use wolfram alpha to help you visualize these equations (it will also give you the names of the corresponding surfaces).*

Problem 50. *The level set of the function* $f(x, y, z) = x^2 + y^2 - z^2$ *corresponding to the value* 0 *is called an* infinite cone. *Find the intersections of this cone with the following planes, and describe them:*

(1) $z = 1$
(2) $x = 1$
(3) $z = y + 1$
(4) $z = \frac{x}{2} + 1$

Because these curves arise as sections of a cone, they are commonly known as conic sections.

Section 2.2

Problem 51. *Show that the limit of the following function along any line through the origin (i.e. $y = mx$ for m any real number, or $x = 0$), as $(x, y) \to (0, 0)$ is always 0.*

$$f(x, y) = \frac{x^3 y}{x^6 + y^2}$$

Can we conclude that the limit exists and is equal to 0?

Problem 52. *Show that the following limit does not exist.*

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^8 + y^2}.$$

Problem 53. *Find, using polar coordinates, the limit as (x, y) approach the origin of:*

$$f(x, y) = \frac{x^2 y^3}{x^2 + y^2}$$

Problem 54. *Compute the following limit by turning it into a single variable function limit.*

$$\lim_{(x,y)\to(1,1)} \frac{\ln(e^x y) - 1}{e^x y - e}.$$

Problem 55. *Compute the following limit:*

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy} - 1}{y}.$$

Hint: *modify the function so that it becomes a limit of a single variable function.*

Problem 56. *Show that the following limit does not exist.*

$$\lim_{(x,y)\to(0,0)} \frac{\sin(2x) + y}{x + y}.$$

Problem 57. *Let*

$$f(x, y) = \begin{cases} \frac{xy^3}{x^2+y^6} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(1) *Compute the limit as $(x, y) \to (0, 0)$ of f along the path $x = 0$.*
(2) *Compute the limit as $(x, y) \to (0, 0)$ of f along the path $x = y^3$.*
(3) *Is f continuous at $(0, 0)$?*

Problem 58. (1) *Can $f(x, y) = \frac{\sin(x+y)}{x+y}$ be made continuous by suitably defining at $(0, 0)$?*
(2) *Can $g(x, y) = \frac{xy}{x^2+y^2}$ be made continuous by suitably defining at $(0, 0)$?*

Section 2.3

Problem 59. *Find the following partial derivatives.*

(a) $w(x, y) = 7xe^{x^2+y^2}$.

$\frac{\partial w}{\partial x} =$

$\frac{\partial w}{\partial y} =$

(b) $f(x,y,z) = \ln(x^2+y^2+z^2)$.

$f_x =$

$f_z =$

Problem 60. *What is the gradient of the function* $f(x,y,z) = xy + \sin z$ *at the point* $(1,1,1)$*?*

Problem 61. *What is the gradient of the function* $f(x,y,z) = x^2 + e^{yz}$ *at the point* $(1,0,0)$*?*

Problem 62. *Compute the gradients of the following functions.*

- $f(x,y,z) = xe^{x^2+y^2+z^2}$
- $g(x,y,z) = z^2 e^x \cos y$
- $h(x,y,z) = \dfrac{\ln(x+z)}{\sin(y^2)}$

Problem 63. *Compute the Jacobian matrix* Df *(i.e. the matrix of partial derivatives).*

$$f(x,y,z) = (z^2 e^x \cos y, xe^{x^2+y^2+z^2})$$

Section 3.1

Problem 64. *Find all the second partial derivatives of* $f(x,y) = e^{-xy^2} + yx^3$*. Write the Hessian of the function (i.e. the matrix of second derivatives).*

Problem 65. *Find all the second partial derivatives of* $f(x,y) = \cos(\sqrt{x^2+y^2})$*. Write the Hessian of* f*.*

Problem 66. *Find* f_{xyz} *for the function*

$$f(x,y,z) = x^4y^3 + x\sin(zy).$$

Problem 67. *Compute the following partial derivatives of the function* $f(x,t) = \sin(x-ct)$*, where* $c \in \mathbb{R}$ *is a constant.*

- $f_x =$
- $f_t =$
- $f_{xx} =$
- $f_{xt} =$
- $f_{tx} =$
- $f_{tt} =$

Why is $f_{tx} = f_{xt}$*? Is this always true?*

Problem 68. *Compute the following partial derivatives of the function* $f(x,t) = \sin(x)\sin(ct)$*, where* $c \in \mathbb{R}$ *is a constant.*

- $f_x =$
- $f_t =$
- $f_{xx} =$
- $f_{xt} =$
- $f_{tx} =$
- $f_{tt} =$

Why is $f_{tx} = f_{xt}$*? Is this always true?*

Problem 69. *Consider the function:*

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

(1) *In which points could it be that* $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$*?*
(2) *Find* $\frac{\partial f}{\partial x}(x,y)$ *and* $\frac{\partial f}{\partial y}(x,y)$ *for* $(x,y) \neq (0,0)$*.*
(3) *Compute the following limits to find* $\frac{\partial f}{\partial x}(0,0)$ *and* $\frac{\partial f}{\partial y}(0,0)$*:*

$$\frac{\partial f}{\partial x}(0,0) := \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$\frac{\partial f}{\partial y}(0,0) := \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h}$$

(4) *Compute the following limits to find* $\frac{\partial^2 f}{\partial x \partial y}(0,0)$ *and* $\frac{\partial^2 f}{\partial y \partial x}(0,0)$*:*

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) := \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h}$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) := \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h}$$

This is an example of a function for which the mixed partial derivatives do not agree.

Section 2.4

Problem 70. *A fly moves along the spiral-shaped curve*

$$c(t) = (\cos t, \sin t, t)$$

(1) *What is its velocity vector?*
(2) *What is its speed? Does it depend on* t*?*
(3) *What is the angle between the trajectory of the fly and the* z*-axis? Does it depend on* t*?* **Hint:** *this is equivalent to finding the angle between the* z*-axis and the velocity vector.*

Problem 71. *Suppose that a particle following the given path* $c(t)$ *flies off on a tangent at* $t = t_0$*. Compute the position of the particle at the given time* t_1*.*

$$c(t) = (\sin t, \cos t, 2t),$$

where $t_0 = \pi$*,* $t_1 = 2\pi$*.*

Problem 72. *A ship moves along the circle*

$$c(t) = (\cos t, \sin t)$$

Meanwhile an iceberg starts at the position $(0,0)$ *and floats in a straight line with velocity vector* $(-1/\pi, 0)$*. Will the iceberg hit the ship? If so, when?*

Problem 73. *Consider the spiral given by* $c(t) = (e^t \cos(t), e^t \sin(t))$*. What is the angle between* c *and* c'*? Does it depend on* t*?*

Problem 74. *A particle under the action of gravity follows the ballistic trajectory*

$$\mathbf{c}(t) = (x_0 + v_{x0}t, y_0 + v_{y0}t - \frac{g}{2}t^2),$$

where $x_0, y_0, v_{x0}, v_{y0}, g \in \mathbb{R}$ *are constants. According to Newton's second law of motion, the acceleration* $\mathbf{c}''(t)$ *of this particle is related to the force* $\mathbf{F}(t)$ *acting on it by the formula* $\mathbf{F} = m\mathbf{c}''$*, where* $m \in \mathbb{R}$ *is a constant (called the inertial mass of the particle). Find* $\mathbf{F}(t)$*.*

Section 2.5

Problem 75. *Suppose that* $T(x,y,z) = \frac{1}{1+x^2+y^2+z^2}$ *describes the temperature distribution at a point* (x,y,z) *in space. Consider the path followed by the fly in problem* 70*:*

$$\mathbf{c}(t) = (\cos t, \sin t, t)$$

(1) *What is the physical interpretation of* $T \circ \mathbf{c}(t)$*?*
(2) *What is* $\frac{d(T\circ \mathbf{c})}{dt}(t)$*? What is its physical interpretation?*
(3) *What is* $\nabla T(x,y,z)$*? What is its physical interpretation?*
(4) *What is* $\mathbf{c}'(t)$*? What is its physical interpretation?*
(5) *What is the physical interpretation of* $\nabla T(\mathbf{c}(t))$*?*
(6) *What is* $\nabla T(\mathbf{c}(t)) \cdot \mathbf{c}'(t)$*? What is its physical interpretation?*

Problem 76. *Suppose* $f(x,y) = (x+y)^7$ *and consider the curve* $c(t) = (t+1, t^2+1)$*. Compute*

$$\frac{d}{dt} f(c(t))$$

in two ways: directly, and using the chain rule.

Problem 77. *Suppose* $f(x,y) = x^2y^2$ *and consider the curve* $c(t) = (\cos t, \sin t)$*. Compute*

$$\frac{d}{dt} f(c(t))$$

in two ways: directly, and using the chain rule.

Problem 78. *Let* $f(u,v) = e^{2v} + \sin(u) : \mathbb{R}^2 \to \mathbb{R}$ *and* $T(s,t) = (s^3+t^3, \ln(s+t)) : \mathbb{R}^2 \to \mathbb{R}^2$*.*

(a) *Compute the Jacobian matrix* $\mathbf{D}f$*.*
(b) *Compute the Jacobian matrix* $\mathbf{D}T$*.*
(c) *Evaluate the Jacobian matrix* $\mathbf{D}(f\circ T)(0,1)$ *using chain rule.*
Hint: *figure out what are the values of* (s,t) *and* (u,v)*.*

(d) *Read the value of* $\frac{\partial(f\circ T)}{\partial s}(0,1)$ *from part (c).*

Section 2.6

Problem 79. *What is the directional derivative of the function* $f(x,y) = 2x + y^3$ *at the point* $p = (2,1)$ *in the direction* $\mathbf{v} = (3,1)$*?*

Problem 80. *What is the directional derivative of the function* $f(x,y,z) = x^2+y^2+z^2$ *at the point* $x = (1,1,1)$ *in the direction* $\mathbf{v} = (2,2,1)$*?*

Problem 81. *Find an equation for the line tangent to the level curve* $x^2 + 2y^2 = 6$ *at the point* $(2,1)$*.*

Problem 82. *Find an equation for the plane tangent to the surface* $x^2 + e^{yz} = e + 1$ *at the point* $(1,1,1)$*.*

Problem 83. *Let* $f(x,y,z) = x + y$ *and consider the surface* $S : yx^2 + yz^2 = 2$*.*

(a) *Verify that* $(1,1,1)$ *is a point in the surface* S*.*
(b) *Find a normal vector to the surface* S *at* $(1,1,1)$*.*
(c) *Find the directional derivative of* f *along the normal direction of* S *at* $(1,1,1)$ *that you found in part (b).*

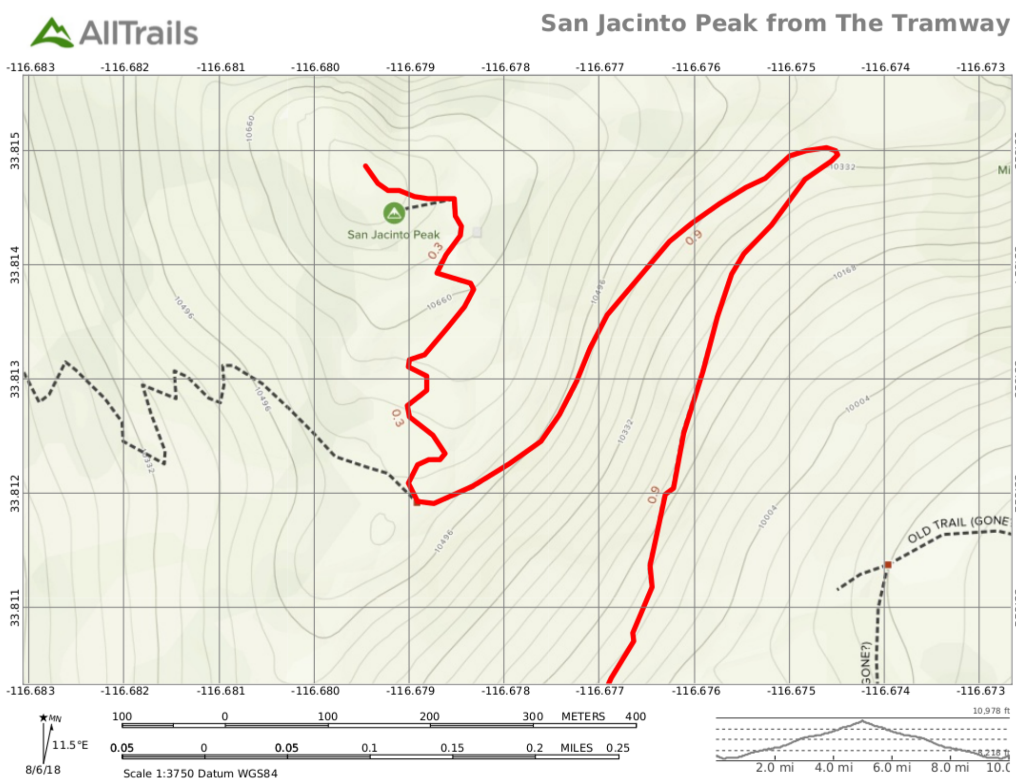
Problem 84. *Let* $f(x,y,z) = 5xy + 5yz + 5xz$ *and* $P = (1,1,1)$ *be a point in space.*

(a) *What is the direction of fastest increase of f at P?*
(b) *What is the maximum value of the rate of change of f at P?*

Problem 85. *You are walking on the graph of $f(x, y) = y\cos(\pi x) - x\cos(\pi y) + 10$, standing at the point $(2, 1, 13)$. Find an x, y-direction you should walk in to stay at the same level.*

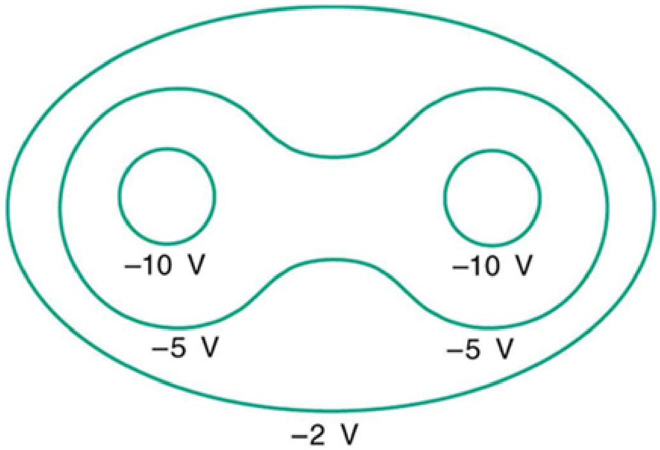
Problem 86. *The picture below shows a map of the last stage of the hike to Mount San Jacinto from the Tramway station (Palm Springs, CA). The red line is the recommended path to follow, and the grey lines are level sets of the height function $h : \mathbb{R}^2 \to \mathbb{R}$ that assigns the height above sea level at each point in the map (these are commonly called contour lines).*

(a) *Draw the gradient vector of h at each intersection of the path (red line) with a contour line.*
(b) *Circle the areas where the path follows the direction of maximum slope.*



Problem 87. *In electromagnetism, the electric field $\mathbf{E} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field (i.e. vector valued function from $\mathbb{R}^3$ to itself), that indicates the electric force received by a particle of positive unit charge. The electric potential is a function $V : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{E} = -\nabla V$ (the negative sign is a historical convention: the electric field goes from bigger potentials, to smaller potentials). In the picture below, there are represented several equipotential lines (i.e. level curves of the function V); the picture represents a situation where two equal negative charges (like electrons) create an electric field.*

(1) *Draw ∇V.*
(2) *Draw $\mathbf{E}$.*



Source:
https://courses.lumenlearning.com/physics/chapter/19-4-equipotential-lines/