PDE Qual Prep Seminar
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207A MATERIAL

ODEs.

Problem 1. *Suppose $f(t,y) : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is continuous on $\mathcal{R}$: $t_0 \le t \le t_0 + a$, $|y - y_0| \le b$, and uniformly Lipschitz continuous with respect to u in $\mathcal{R}$. Let M be an upper bound for $|f(t,y)|$ on $\mathcal{R}$ and $\alpha := \min(a, b/M)$. Prove that*

$$\frac{dy}{dt} = f(t,y) \quad y(t_0) = y_0$$

has a solution on $[t_0, t_0 + \alpha]$.

[Source: HW1 2015, 2017, Midterms 2015, 2017, A. Moradifam]

Laplace's equation.

Problem 2. *Prove that if $u \in \mathcal{C}^2(\Omega)$ is harmonic, then*

$$u(x) = \fint_{\partial B(x,r)} u dS,$$

for all $B(x,r) \subset \Omega$.

[Source: Midterm 2015, A. Moradifam]

Problem 3. *(Harnack's Inequality) Let $V \subset\subset \Omega$. Prove that there exists a constant C, depending only on V, such that*

$$\sup_V u \le C \inf_V u$$

for all nonnegative harmonic functions u in Ω.

[Source: Midterm 2017, A. Moradifam]

Problem 4. *Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ solve the Laplace equation*

$$-\Delta u = f \text{ on } \Omega,$$

with $u = g$ on $\partial\Omega$. Prove that there exists a constant C, independent of u, such that

$$\max_{\Omega} |u| \le C(\max_{\partial\Omega} |g| + \max_{\Omega} |f|).$$

[Source: HW3 2015, 2017 (Evans Ch. 2), Midterms 2015, 2017, A. Moradifam]

Problem 5. *Let $u_1, u_2 \in \mathcal{C}^2(\Omega)$ solve the Laplace equation*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases}$$

Prove that $u_2 - u_1 = c$, where c is a constant.

[Source: Final 2017, A. Moradifam]

Problem 6. *Let*

$$\mathcal{A} := \{\omega \in \mathcal{C}^2(\bar{\Omega}) \colon \omega = g \text{ on } \partial\Omega\}$$

and define $I : \mathcal{A} \to \mathbb{R}$ *by*

$$I(\omega) := \int_\Omega \frac{|D\omega|^2}{2} - \omega f dx.$$

Prove that $u \in \mathcal{A}$ *is a minimizer of* I *in* $\mathcal{A}$ *if and only if* u *solves the Laplace equation*

$$-\Delta u = f$$

[Source: Final 2015, Midterm 2017, A. Moradifam]

Problem 7. *Suppose* $u \in \mathcal{C}^2(\bar{\Omega})$ *solves the equation*

$$\begin{cases} \Delta u = g, & \text{in } \Omega \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where ν *is the outward unit normal vector on* $\partial\Omega$*. Prove that*

$$\int_\Omega g\varphi dx = 0$$

for every harmonic function $\varphi \in \mathcal{C}(\bar{\Omega})$.

[Source: Midterm 2017, A. Moradifam]

Problem 8. *Let* $(u_n)_{n=1}^\infty$ *be a sequence of harmonic functions defined on an open bounded subset* U *of* $\mathbb{R}^d$, $d \geq 2$*. Assume that* $u_n \to u$ *uniformly on* U*. Prove that* u *is harmonic on* U.

[Source: Qual 2015, Final 2014, J. Kelliher]

Heat equation.

Problem 9. *Let* $u \in \mathcal{C}_1^2(\Omega_T)$ *solve the heat equation. Prove that*

$$u(x,t) = \frac{1}{4r^n} \int\int_{E(x,t;r)} u(y,s) e^{-\frac{|x-y|^2}{(t-s)^2}} dyds,$$

for every heat ball $E(x,t;r) \in \Omega_T$*. Note that*

$$E(x,t;r) = \{(y,s) \in \mathbb{R}^{n+1} \colon s \leq t, \frac{1}{(4\pi(t-s))^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \geq \frac{1}{r^n}\}$$

and

$$\frac{1}{r^n} \int\int_{E(r)} \frac{|y|^2}{s^2} dyds = 4$$

where $E(r) = E(0,0;r)$.

[Source: Final 2017, A. Moradifam]

Problem 10. *(Strong maximum principle for the heat equation) Let* $\Omega \subset \mathbb{R}^n$ *and* $T > 0$ *and suppose* $u \in \mathcal{C}_\infty^\in(\Omega_T) \cap \mathcal{C}(\bar{\Omega_T})$ *solves the heat equation in* Ω_T*. Prove that if* Ω *is connected and there exists* $(x_0, t_0) \in \Omega_T$ *such that*

$$u(x_0, t_0) = \max_{\bar{\Omega_T}} u,$$

then u *is constant in* $\bar{\Omega_{t_0}}$. **Hint:** *Use the mean-value property for the heat equation:*

$$u(x,t) = \frac{1}{4r^n} \int\int_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds,$$

where $E(x,t;r)$ is the heat ball centered at (x,t).

[Source: Final 2015, A. Moradifam]

Problem 11. *Let $g \in \mathcal{C}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and define*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy$$

where Φ is the fundamental solution of the heat equation ($\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{\frac{-}{|x|^2}4t}$). Prove that u solves the heat equation and

$$\lim_{(x,t)\to(x_0,0^+)} u(x,t) = g(x_0),$$

for all $x_0 \in \mathbb{R}^n$.

[Source: Midterm 2015, Final 2017, A. Moradifam]

Problem 12. *(Backwards Uniqueness for the heat equation) Suppose $u, \tilde{u} \in \mathcal{C}^2(\bar{U}_T)$ solve the heat equation*

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega \times (0,T] \\ u = g & \text{on } \partial\Omega \times [0,T], \end{cases}$$

Prove that if $u(x,T) = \tilde{u}(x,T)$ $(x \in \Omega)$, then

$$u = \tilde{u} \text{ in } \Omega \times (0,T].$$

[Source: Qual 2016, A. Moradifam]

Problem 13. *Let $u(x,t) = v\left(\frac{x^2}{t}\right)$. (a) Show that*

$$u_t - u_{xx} = 0$$

if and only if

$$4z\frac{\partial^2 v}{\partial z^2} + (2+z)\frac{\partial v}{\partial z}(z) = 0, \;\; z > 0.$$

(b) Use part (a) to obtain the fundamental solution of the heat equation in dimension $n = 1$.

[Source: HW4 2015, 2017 (modified from Evans Ch. 2), Final 2015, A. Moradifam]

Wave equation.

Problem 14. *(a) Show that the general solution of the PDE $u_{xy} = 0$ is*

$$u(x,y) = F(x) + G(y)$$

for arbitrary functions F and G.
(b) Using the change of variable $\xi = x+t$, $\eta = x-t$, show that $u_{\xi\eta} = 0$ if and only if

$$u_{tt} - u_{xx} = 0.$$

(c) Use (a) and (b) to prove that the general solution of the wave equation in dimension one is given by

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy$$

[Source: Final 2015, Final 2017 (Evans Ch. 2), A. Moradifam]

Problem 15. *(Finite speed of propagation for solutions of the wave equation) Suppose $u \in \mathcal{C}^2(\mathbb{R}^n \times (0,\infty))$ solves the wave equation. Prove that if $u = u_t = 0$ on $B(x_0, t_0) \times \{0\}$, then $u \equiv 0$ within the cone*

$$C = \{(x,t) \colon 0 \le t \le t_0,\ |x - x_0| \le t_0 - t\}$$

Hint: *consider the energy functional*

$$e(t) = \frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2(x,t) + |Du(x,t)|^2 dx \quad (0 \le t \le t_0)$$

[Source: Final 2015, Qual 2016, Final 2017, A. Moradifam]

Problem 16. *Assume that u is a smooth solution to the wave equation in $\mathbb{R}^d$, $d \ge 1$, in the form*

$$\partial_{tt} u = \Delta u.$$

Define the energy-density $e(t,x) := \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2$.

(a) *Show that for $d = 1$, e itself satisfies the one-dimensional wave equation.*
(b) *Show that e does not, in general, satisfy the d-dimensional wave equation for $d \ge 2$.*
(c) *Suppose that $u \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^d)$, $d \ge 1$, and compactly supported in space for all t. Show that*

$$\int_{\mathbb{R}^d} (\partial_{tt} e(t,x) - \Delta e(t,x)) dx = 0$$

[Source: Qual 2015, J. Kelliher]

Other equations.

Problem 17. *Write down an explicit formula for a function u solving the equation*

$$\begin{cases} u_t - b \cdot \nabla u + cu = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constant.

[Source: HW2 2015, 2017 (Evans Ch.2), Qual 2016, A. Moradifam]

Problem 18. *Suppose u is an integral solution of*

$$u_t + F(u)_x = 0,$$

and $u \in \mathcal{C}^1((\mathbb{R} \times (0,\infty)) \setminus \Gamma)$, where Γ is a one dimensional curve. Prove that if u is discontinuous on Γ, then it should satisfy the jump condition

$$(F(u_l) - F(u_r))\eta^1 + (u_l - u_r)\eta^2 = 0,$$

where $(\eta^1, \eta^2)(x)$ is the normal to the curve Γ at x.

[Source: Final 2015 (Evans 3.4.1), A. Moradifam]

Flow maps.

Problem 19. *Let $\mathbf{u}(t,x)$ be a time-varying vector field on $\mathbb{R}^d$.*

(a) *Define what it means for $X : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ to be a flow map for $\mathbf{u}$.*

(b) *Assume that $\mathbf{u}_1$, $\mathbf{u}_2$ are Lipschitz continuous vector fields with Lipschitz constants M_1, M_2 for all $t \in \mathbb{R}$; that is, $|\mathbf{u}_j(t,x) - \mathbf{u}_j(t,y)| \le M_j|x-y|$, $j = 1,2$, for all $t \in \mathbb{R}$, $x, y \in \mathbb{R}^d$. Let X be the flow map for $\mathbf{u} := \mathbf{u}_1 + \mathbf{u}_2$ and X_j be the flow map for $\mathbf{u}_j$, $j = 1,2$ (you do not need to prove that such flow maps exist and are unique). Show that*

$$|X(t,x) - X_1(t,x)| \le M_2 t e^{M_1 t}.$$

[Source: Qual 2016, J. Kelliher]

Problem 20. *Let $\mathbf{v}(t,x)$ be a time-varying vector field on $\mathbb{R}^d$.*

(a) *Define what it means for $X : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ to be a flow map for $\mathbf{v}$.*

(b) *Suppose that $\mathbf{v}$ is uniformly Lipschitz-continuous in space; that is,*

$$|\mathbf{v}(t,x) - \mathbf{v}(t,y)| \le M|x-y| \text{ for all } x, y \in \mathbb{R}^d$$

for some $M > 0$. Prove that there exists a flow map for $\mathbf{v}$.

(c) *Prove that the flow map for $\mathbf{v}$ is unique.*

[Source: Qual 2015, J. Kelliher]

Problem 21. *Let $A(t)$ be a continuous function from t in $\mathbb{R}$ to the space of square, real-valued matrices.*

(a) *Show that for every solution of the (non-autonomous) linear system, $\dot{\mathbf{x}} = A(t)\mathbf{x}$, we have*

$$\|\mathbf{x}(t)\| \le \|\mathbf{x}(0)\| e^{\int_0^t \|A(s)\| ds},$$

where $\|A(s)\|$ is the operator norm and $\|\mathbf{x}(t)\|$ is the usual Euclidean norm.

(b) *Show that if $\int_0^t \|A(s)\| ds < \infty$ then every solution, $\mathbf{x}(t)$, has a finite limit as $t \to \infty$.*

[Source: Final 2014, J. Kelliher]

Method of characteristics.

Problem 22. *In this problem, we will seek a solution in the upper half plane to*

$$\begin{cases} -y\partial_x u + x\partial_y u = au & \text{for } x \in \mathbb{R}, y \ge 0, \\ u(x,0) = \psi(x) & \text{for } x \ge 0, \end{cases}$$

where a is a real constant and $\psi : [0,\infty) \to \mathbb{R}$ is differentiable with $\psi(0) = \psi'(0) = 0$.

(a) *What are the characteristic equations for this PDE?*

(b) *Solve the characteristic equations.*

(c) *Use the solution in (b) to the characteristic equations to obtain an explicit solution to the PDE in the form $u = u(x,y)$.*

(d) *Show that any compactly supported, differentiable solution to our PDE is unique whenever $a \le 0$.* **Hint:** *rewrite the PDE in the form $\mathbf{x}^\perp \cdot \nabla u = au$, where $\mathbf{x}^\perp = (-y,x)$, and make an energy argument. (The fact that $\operatorname{div} \mathbf{x}^\perp = 0$ will be very useful).*

[Source: Midterm 2018, J. Kelliher]

Problem 23. *Consider Burger's equation on the half-line, $r \in (0,\infty)$, with a non-linear forcing term:*

$$\begin{cases} \partial_t v + v\partial_r v = -\frac{v^2}{r} & \text{for } t \in (0,\infty),\ r \in (0,\infty), \\ v = v_0 & \text{for } t = 0,\ r \in (0,\infty). \end{cases}$$

Assume that $v_0 \in \mathcal{C}^\infty[0,\infty)$ with $v_0(0) = 0$ [note this implies that $\lim_{r\to 0^+} v^2(r)/r = 0$].

In this problem, you will be asked to use the method of characteristics. As in any such problem, it is always possible that the characteristic projections will collide at some time, $T_1 \geq 0$, meaning that if $T \geq 0$ is the time of existence of a classical solution, then $T \leq T_1$.

But because we are working on a half-line, a classical solution can also cease to exist because a characteristic projection reaches $r = 0$ for some $t = T_2 > 0$, taking it outside the domain of solution. This means that also $T \leq T_2$, and hence $T = \min\{T_1, T_2\}$.

(a) *Show that if we parametrize each characteristic by time and write each characteristic in the form $t \mapsto (r(t), v(r(t)))$ with $r(0) = r_0$, then:*

$$r(t)^2 = r_0^2 + 2r_0 v_0(r_0)t,$$
$$v(t, r(t)) = \frac{r_0 v_0(r_0)}{r(t)}$$

(b) *Show that*

$$T_1 = \inf_{r_0 > s_0 > 0} \frac{r_0^2 - s_0^2}{2(s_0 v_0(s_0) - r_0 v_0(r_0)}$$

(c) *Show that*

$$T_2 = \inf_{r_0 > 0} \left(-\frac{r_0}{2v_0(r_0)} \right)$$

[Source: Midterm 2018, J. Kelliher]

Problem 24. (a) *In the plane, solve the equation $y\frac{\partial u}{\partial x} - 4x\frac{\partial u}{\partial y} = 0$ with the condition that $u(0,y) = f(y)$ for all $y \in \mathbb{R}$. Assume that $f \in \mathcal{C}^\infty(\mathbb{R})$ and find only $\mathcal{C}^\infty(\mathbb{R}^2)$ solutions.*

(b) *Is the problem solvable uniquely in the full plane for all $f \in \mathcal{C}^\infty(\mathbb{R}^2)$? If not, what additional property or properties of f is or are required for the problem to be solvable in the full plane? (Remember that we seek only $\mathcal{C}^\infty$ solutions).*

[Source: Qual 2016, J. Kelliher]

Problem 25. *Let $\mathbf{v}(t,x)$ be as in problem 20 above in part (b), and suppose that $\rho = \rho(t,x)$ solves*

$$\partial_t \rho + \mathbf{v} \cdot \nabla \rho = \rho^2,$$
$$\rho(0) = \rho_0,$$

where ρ_0 is continuous and bounded on $\mathbb{R}^d$. Suppose that $\rho_0(x) \leq M$ for all $x \in \mathbb{R}^d$. Express the maximum possible time of existence of ρ in terms of M. (You need not actually prove existence however.)

Hint: *Use the result of the problem 20 above - you can do so even if you didn't solve it.*

[Source: Qual 2015, J. Kelliher]

Fourier Transform.

Problem 26. *Solve the following initial value problem using the Fourier transform:*

$$\begin{cases} \partial_t tu + c^2 \partial_{xxxx} u = 0 & \text{in } (0,\infty) \times \mathbb{R}, \\ u = f, \partial_t u = g & \text{on } \{0\} \times \mathbb{R}\,. \end{cases}$$

Give your answer in the form of an inverse Fourier transform.

[Source: Qual 2015, Final 2018, J. Kelliher]

Sobolev spaces.

Problem 27. *Fix $d \geq 1$, let U be an open subset of $\mathbb{R}^d$, and let $p \in [1,\infty]$. For any $u \in W^{1,p}(\mathbb{R}^d)$ let $R(u) = u_{|_U}$. Show that this defines a continuous linear operator from $W^{1,p}(\mathbb{R}^d)$ to $W^{1,p}(U)$.*

[Source: Final 2018, J. Kelliher]

Problem 28. *Let $U \subseteq \mathbb{R}^d$ be open.*

(a) *Integrate by parts to prove the interpolation inequality,*

$$\|\nabla u\|_{L^2} \leq \|u\|_{L^2}^{1/2} \|\Delta u\|_{L^2}^{1/2}$$

for all $u \in \mathcal{C}_c^\infty(U)$.

(b) *Assume now that U is bounded with smooth boundary. Prove that the same inequality as in (a) holds for all $u \in H^2(U) \cap H_0^1(U)$.*

[Source: Final 2018, J. Kelliher]

Problem 29. *Let U be a bounded, open set in $\mathbb{R}^n$ with smooth boundary. Show that*

$$\|Du\|_{L^2(U)}^2 \leq C\|u\|_{L^2(U)} \|D^2 u\|_{L^2(U)} \text{ for all } u \in H_0^2(U)$$

where C does not depend on u.

[Source: Midterm 2015, J. Kelliher]

Problem 30. *Prove or disprove the following:*

Let U be a bounded, open subset of $\mathbb{R}^2$. If $u \in W^{1,2}(U)$, then $u \in L^\infty(U)$ with the estimate

$$\|u\|_{L^\infty(U)} \leq C\|u\|_{W^{1,2}(U)}$$

where C does not depend on u.

[Source: Midterm 2015, J. Kelliher]

Problem 31. *Let U be the open unit ball in $\mathbb{R}^d$.*

(a) *Let*

$$u(x) = |x|^{-\alpha}.$$

Determine which values of $\alpha > 0$, $d \geq 1$, and $p > 1$ give u in $W^{1,p}(U)$.

(b) *Show that*

$$u(x) = \log\log\left(1 + |x|^{-1}\right)$$

belongs to $W^{1,2}(U)$ but does not belong to $L^\infty(U)$.

[Source: Qual 2016, J. Kelliher]

Problem 32. (a) *Let u be a function on $\mathbb{R}^d$. For any $h \in \mathbb{R}^d$, define $\tau_h u = u(\cdot - h)$: this is translation of u by h. Show that*

$$\|\tau_h u - u\|_{L^p} \leq |h| \|\nabla u\|_{L^p}$$

for any $1 \leq p < \infty$.

Suggestion: *For $1 \leq p < \infty$, use the fundamental theorem of calculus trick/technique.*

(b) *Now assume that $p \in (d, \infty]$. Take as a given the following lemma:*

Lemma: *For all $p \in (d, \infty]$,*

$$\|u\|_\infty \leq C_1 \|u\|_{L^p} + C_2 \|\nabla u\|_{L^p},$$

where C_1 and C_2 depend on p and d.

Use this lemma to show that for all $u \in W^{1,p}(\mathbb{R}^d)$,

$$\|u\|_{L^\infty} \leq C \|u\|_{L^p}^{1-\theta} \|\nabla u\|_{L^p}^{\theta},$$

where $\theta = d/p$. **Suggestion:** *First make a scaling argument to show that*

$$\|u\|_{L^\infty} \leq C_1 \lambda^{\frac{d}{p}} \|u\|_{L^p} + C_2 \lambda^{-1+\frac{d}{p}} \|\nabla u\|_{L^p}.$$

for any $\lambda > 0$. (That is, apply the lemma to $u(\cdot/\lambda)$). Then choose the optimal λ to obtain the result.

(c) *Conclude from (a) and (b) that for all $u \in W^{1,p}(\mathbb{R}^d)$, $p \in (d, \infty]$,*

$$|u(x) - u(y)| \leq C|x - y|^\gamma \|\nabla u\|_{L^p}$$

for $\gamma = 1 - d/p$.

Note: *This provides an alternative proof of Morrey's inequality (given a proof of the lemma above).*

[Source: Final 2016, J. Kelliher]

Others.

Problem 33. *Using separation of variables, solve the wave equation with homogeneous Dirichlet boundary conditions on $(t, x) \in \mathbb{R} \times (0, L)$,*

$$\begin{cases} \partial_{tt} u = c^2 \partial_{xx} u, \\ u(t, 0) = u(t, L) = 0, \\ u(0, x) = f(x), \\ \partial_t u(0, x) = g(x). \end{cases}$$

You can assume that $f, g \in \mathcal{C}^\infty((0, L))$.

[Source: Final 2018, J. Kelliher]

Problem 34. *Consider the one-dimensional conservation law,*

$$\begin{cases} \partial_t u + \partial_x F(u) = 0, \\ u(0) = g, \end{cases}$$

with initial data

$$g(x) = \begin{cases} u_l & \text{if } x < 0, \\ u_r & \text{if } x > 0. \end{cases}$$

This is called the Riemann problem. *The solution is to be on* $(0, \infty) \times \mathbb{R}$ *(time* $\times$ *space). We assume that* F *is smooth and uniformly strictly convex (that is,* $F'' > c > 0$ *for some constant* c*) and that* u_l *and* u_r *are constants.*

(a) *Define what it means for* u *to be an entropy solution to the one-dimensional conservation law.*

(b) *Show that if* $u_l > u_r$ *then*

$$u(t,x) = \begin{cases} u_l & \text{if } \frac{x}{t} < \sigma, \\ u_r & \text{if } \frac{x}{t} > \sigma \end{cases}$$

is an entropy solution to Riemann's problem, where

$$\sigma := \frac{F(u_l) - F(u_r)}{u_l - u_r}.$$

[Source: Qual 2015, J. Kelliher]

207C MATERIAL

Maximum Principles for elliptic and parabolic operators.

Problem 35. *Suppose* u *is a smooth solution to*

$$\begin{cases} \Delta u = u^3 + u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

Show that $u \equiv 0$ *in* U *using maximum principle.*

[Source: Final 2018, P.N. Chen]

Problem 36. *Prove that there is at most one smooth solution of*

$$\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \times [0, T] \\ u = g & \text{on } U \times \{0\} \end{cases}$$

[Source: Final 2018, HW, P.N. Chen, Evans' Ch 7]

Problem 37. *Suppose* u *is a smooth solution to*

$$\begin{cases} u_t - \Delta u + cu = 0 \\ u = 0 \\ u = g \end{cases}$$

where the function c *satisfies* $c \geq \delta > 0$*. Show that* $|u(x,t)| \leq Ce^{-\gamma t}$*.*

[Source: Final 2018, HW, P.N. Chen, Evans' Ch 7]

Problem 38. *Assume that* u *is a smooth solution of the PDE from the problem above, that* $g \geq 0$*, and that* c *is bounded (but not necessarily nonnegative). Show* $u \geq 0$*.*

Hint: *What PDE does* $v := e^{-\lambda t} u$ *solve?*

[Source: P.N. Chen's HW, Evans' Ch7]

Problem 39. *Assume U is connected. Use (a) energy methods and (b) the maximum principle to show that the only smooth solutions of the Neumann boundary-value problem*

$$\begin{cases} -\Delta u = 0 & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

are $u \equiv C$ for some constant C.

[Source: P.N. Chen's HW, Evan's Ch 6]

Problem 40. *We say that the uniformly elliptic operator*

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu,$$

satisfies the weak maximum principle if for all $u \in \mathcal{C}^2(U) \cap \mathcal{C}(\bar{U})$,

$$\begin{cases} Lu \leq 0 & \text{in } U, \\ u \leq 0 & \text{on } \partial U, \end{cases}$$

implies that $u \leq 0$ on U.

Suppose that there exists a function $v \in \mathcal{C}^2(U) \cap \mathcal{C}(\bar{U})$ such that $Lv \geq 0$ in U and $v > 0$ on $\bar{U}$. Show that L satisfies the weak maximum principle.

Hint: *Find an elliptic operator M such that $w = y/v$ satisfies $Mw \leq 0$ in the region $\{u > 0\}$. To do this, first compute $(v^2 w_{x_i})_{x_j}$.*

[Source: Qual 2016, J. Kelliher, P.N. Chen's HW, Evan's Ch6]

Problem 41. *Let Lu be defined as:*

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x,t) u_{x_i x_j} + \sum_{i=1}^{n} b^i(x,t) u_{x_i} + c(x,t)u.$$

Suppose $u \in \mathcal{C}_1^2(U_T) \cap \mathcal{C}^0(\bar{U_T})$ satisfies

$$u_t + Lu \geq 0, \text{ in } U_T.$$

Assume $c(x,t) \geq m$ with m being a fixed constant (either positive or negative), show that if $u \geq 0$ on Γ_T, then $u \geq 0$ in $\bar{U_T}$.

[Source: Qual 2016, J. Kelliher]

Sobolev Spaces.

Problem 42. *Consider a function $u : (-1,1) \to \mathbb{R}$ given by $u(x) = |x|$.*

a) *Show that u is Lipschitz continuous but not continuously differentiable.*
b) *Show that u is weakly differentiable and find the weak derivative.*
c) *How about the second weak derivative? Find the distributional derivative and weak derivative of u' if possible; if not, explain why.*

[Source: Qual 2015, J. Zhang or J. Kelliher]

Problem 43. *Verify thatt if $n > 1$, the unbounded function $u = \log\log\left(1 + \frac{1}{|x|}\right)$ belongs to $W^{1,n}(U)$, for $U = B^0(0,1)$.*

[Source: Po-Ning's HW, Evans' Ch.5]

Problem 44. *Fix $\alpha > 0$ and let $U = B^0(0,1)$. Show there exists a constant C, depending only on n and α, such that*

$$\int_U u^2 dx \leq C \int_U |Du|^2 dx,$$

provided

$$|\{x \in U \colon u(x) = 0\}| \geq \alpha, \ \ u \in H^1(U)$$

[Source: Po-Ning's HW, Evans' Ch.5]

Problem 45. *(Variant of Hardy's inequality) Show that for each $n \geq 3$ there exists a constant C so that*

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq C \int_{\mathbb{R}^n} |Du|^2 dx$$

for all $u \in H^1(\mathbb{R}^n)$.

Hint: $|Du + \lambda \frac{x}{|x|^2} u|^2 \geq 0$ *for each* $\lambda \in \mathbb{R}$.

[Source: Po-Ning's HW, Evans' Ch.5]

Existence of solutions.

Problem 46. *Let*

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + cu.$$

Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\cdot,\cdot]$ satisfies the hypothesis of the Lax-Milgram Theorem, provided

$$c(x) \geq -\mu \ (x \in U).$$

[Source: P.N. Chen's HW, Evan's Ch 6]

Problem 47. *A function $u \in H_0^2(U)$ is a weak solution of this boundary-value problem for the biharmonic equation*

$$\begin{cases} \Delta^2 u = f & \text{in } U \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

provided

$$\int_U \Delta u \Delta v dx = \int_U f v dx$$

for all $v \in H_0^2(U)$. Given $f \in L^2(U)$, prove that there exists a unique weak solution of this problem.

[Source: P.N. Chen's HW, Evan's Ch 6]

Problem 48. *Let U be a connected, bounded open set in $\mathbb{R}^n$. We say that $u \in H^1(U)$ is weak solution of*

$$\begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

if

$$\int_U Du \cdot Dv dx = \int_U fv dx \text{ for all } v \in H^1(U).$$

Suppose $f \in L^2(U)$. Prove that there exists a weak solution to the problem above if and only if $\int_U f dx = 0$.

[Source: Qual 2015, J. Zhang or J. Kelliher, P.N. Chen's HW, Evan's Ch 6]

Problem 49. *Consider the following Dirichlet problem*

$$\begin{cases} -\Delta u + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

where μ is a given nonzero constant.

a) *Show the existence of a weak solution $u \in H_0^1(U)$ of the problem above for $\mu > 0$.*
b) *Discuss the problem when $\mu < 0$.*

[Source: Qual 2015, J. Zhang or J. Kelliher]

Problem 50. *Assume U is connected. Consider the Poisson's equation with the Neumann boundary-value problem*

$$\begin{cases} -\Delta u = f, & \text{on } U, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial U. \end{cases}$$

a) *State the definition of the weak solution to this problem;*
b) *Show that for $f \in L^2(U)$, the above problem has a weak solution if and only if $\int_U f dx = 0$;*
c) *If $f = 0$, use the energy method to show that the only smooth solutions of this problems are $u = C$, for some constant C.*

[Source: Qual 2016, J. Kelliher]

Others.

Problem 51. *Consider Laplace's equation with potential function c:*

$$-\Delta u + cu = 0, \tag{1}$$

and the divergence structure equation:

$$-\mathrm{div}(\mathrm{aDv}) = 0 \tag{2}$$

where the function a is positive.

a) *Show that if u solves (1) and $w > 0$ also solves (1), then $v := u/w$ solves (2) for $a := w^2$.*
b) *Conversely, show that if v solves (2), then $u := va^{1/2}$ solves (1) for some potential c.*

[Source: P.N. Chen's HW, Evan's Ch 6]