MATH 10A MIDTERM EXAMINATION
Summer Session E
UC Riverside

August 23, 2018

Name: Solution

Instructions:

1. The exam has a total of 100 points. You have 50 minutes.
2. Problem 4 is an Extra Credit question worth 10 points. The points will be added to the final score, up to a maximum of 100. Doing this problem is not required.
3. **Notes, books, calculators or other electronic devices may not be used in this examination.**
4. Show all of your work to receive full credit. You may use any result or theorem done in class.
5. If you have a question or need to go to the restroom, raise your hand and wait for a proctor.
6. Points for each problem are indicated within the problem, and in the table below.

Question	Points	Score
1	40	
2	30	
3	30	
4	10	
Total:	100	

Problem 1. *In meteorology, isobar maps are used to visualize the atmospheric pressure at a given time at sea level. From a mathematical point of view, the atmospheric pressure is a function $p : \mathbb{R}^2 \to \mathbb{R}$ that assigns at each point in the map (represented by the plane $\mathbb{R}^2$) the atmospheric pressure that you would measure at that point at a given time if you were at sea level (that is, a number in $\mathbb{R}$). An isobar is a curve in the map all of whose points have the same pressure, which can be expressed mathematically as all the points (x, y) in $\mathbb{R}^2$ such that $p(x, y) = C$, for some value C in $\mathbb{R}$.*

a) *(5 points) Select from the following list what kind of function is p.*

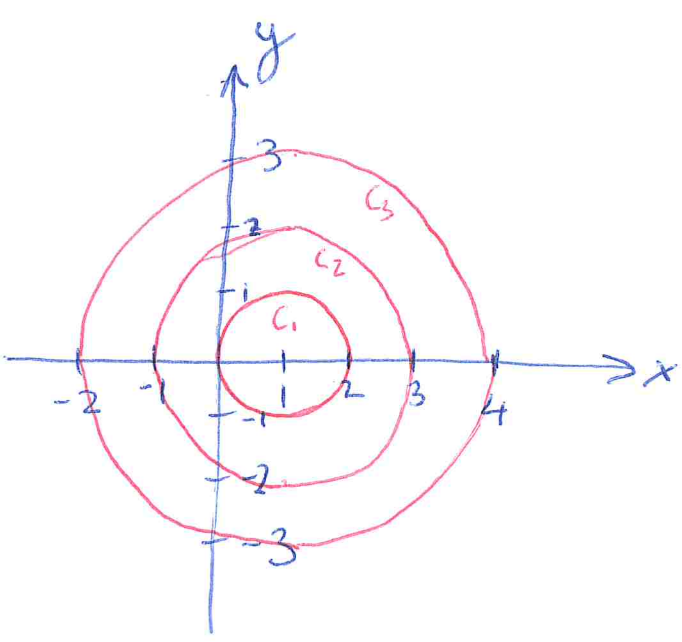
1) *Real-valued function of a single variable.*
2) *Vector-valued function of a single variable.*
3) *Real-valued function of several variables.*
4) *Vector-valued function of several variables.*

b) *(5 points) What is the mathematical name (the name that we used in this class) given to the curves of the form $p(x, y) = C$?*

Level sets / Level curves / Contour lines

Hint: *it's not isobar; isobar is a specific name given to these curves when p represents pressure (from the greek "isos" = equal, "baros" = weight).*

c) *(10 points) If $p(x, y) = 1024 - (x - 1)^2 - y^2$, draw the isobars corresponding to the values of the pressure $C_1 = 1023$, $C_2 = 1020$ and $C_3 = 1015$ in the same graph. Make sure to label your axis.*



$C_1 = 1023$

$1023 = 1024 - (x-1)^2 - y^2$

$\boxed{(x-1)^2 + y^2 = 1}$

$C_2 = 1020$

$1020 = 1024 - (x-1)^2 - y^2$

$\boxed{(x-1)^2 + y^2 = 4}$

$C_3 = 1015$

$1015 = 1024 - (x-1)^2 - y^2$

$\boxed{(x-1)^2 + y^2 = 9}$

d) *(10 points) If $p(x, y) = 1024 - (x-1)^2 - y^2$ as in part c), what is the gradient of p at a point (x, y)?*

$$\nabla p(x,y) = \left(-2(x-1), -2y\right)$$

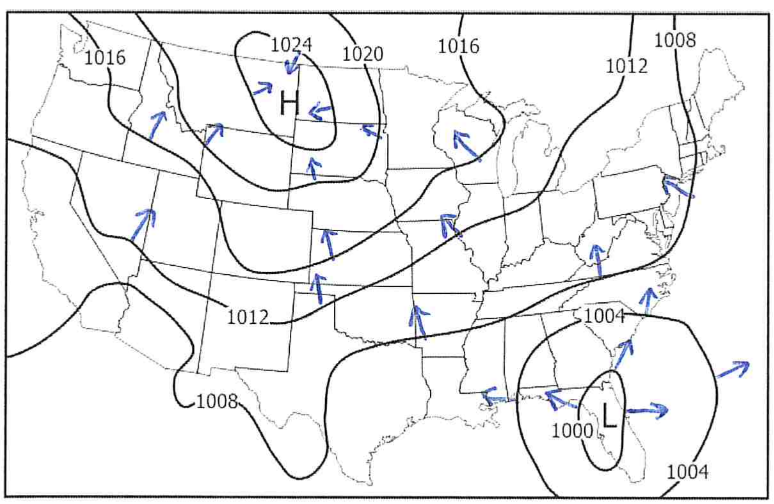
e) *(10 points) If $p(x, y)$ is as in parts c) and d), what is the directional derivative of p at the point $(6, -5)$ in the direction of $\mathbf{v} = (3, 4)$?*

$$\nabla p(6,-5) = \left(-2(6-1), 10\right) = (-10, 10)$$

$$\vec{v} = (3,4) \Rightarrow \|\vec{v}\| = \sqrt{3^2+4^2} = 5, \quad \frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{3}{5}, \frac{4}{5}\right)$$

Directional derivative $= D_{\vec{v}}p(6,-5) = \nabla p(6,-5) \cdot \dfrac{\vec{v}}{\|\vec{v}\|} = (-10,10)\cdot\left(\frac{3}{5},\frac{4}{5}\right) = \boxed{2}$

f) *(5 points) In the following isobar map of the USA, isobars have been labeled by their values (from low pressure L of 1000hPa in Florida to high pressure H of 1024hPa in the Rocky mountains). Draw schematically the direction and orientation of the gradient of p, $\nabla p(x, y)$, at each isobar. Draw at least 3 gradients for each curve.*



Source: `http://megritchie1993.blogspot.com/2014/07/isobar.html`

Problem 2. *Consider the piecewise defined function:*

$$f(x,y,z) = \begin{cases} \frac{x^6yz}{x^{12}+y^4+z^4} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,z) = (0,0,0) \end{cases}$$

a) *(10 points) Find the parametric equation of a line $l(t)$ in $\mathbb{R}^3$ that goes through the origin $(0,0,0)$ and is parallel to the vector $(1,-1,1)$.*

$$l(t) = (0,0,0) + t(1,-1,1) = (t,-t,t)$$

b) *(5 points) What is the limit of $f(x,y,z)$ as you approach the origin along the line $l(t)$ that you found in part a)?* along $l(t) = (t,-t,t)$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^6yz}{x^{12}+y^4+z^4} \stackrel{\swarrow}{=} \lim_{t\to0} \frac{t^6(-t)t}{t^{12}+(-t)^4+t^4} = \lim_{t\to0} \frac{-t^8}{t^{12}+2t^4} = \lim_{t\to0} \frac{-t^4}{t^8+2} = \boxed{0}$$

this function is continuous at $t=0$.

c) *(5 points) Consider the curve $c(t) = (t,t^3,t^3)$. Note that $c(0) = (0,0,0)$, so this curve goes through the origin. What is the limit of $f(x,y,z)$ as you approach the origin along the curve $c(t)$?* along $c(t) = (t,t^3,t^3)$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^6yz}{x^{12}+y^4+z^4} \stackrel{\swarrow}{=} \lim_{t\to0} \frac{t^6 t^3 t^3}{t^{12}+(t^3)^4+(t^3)^4} = \lim_{t\to0} \frac{t^{12}}{t^{12}+t^{12}+t^{12}} = \lim_{t\to0} \frac{\cancel{t^{12}}}{3\cancel{t^{12}}} = \boxed{\frac{1}{3}}$$

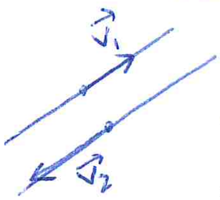
d) *(10 points) Is f continuous at the origin? Justify your answer.*

No, f is not continuous at the origin because $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z)$ does not exist (since when we approached $(0,0,0)$ along 2 paths we got 2 different results).

Problem 3. *Consider the two lines:*

$$l_1(t) = (1,0,1) + t(1,1,-1)$$
$$l_2(s) = (2-2s, -2s, 2s)$$

a) (10 points) What is the angle between the direction vectors of l_1 and l_2? Are the two lines parallel?



Direction vector of l_1: $\vec{v}_1 = (1,1,-1)$ $\quad \|\vec{v}_1\| = \sqrt{3}$

Direction vector of l_2: $\vec{v}_2 = (-2,-2,2)$ $\quad \|\vec{v}_2\| = \sqrt{12} = 2\sqrt{3}$

$\vec{v}_1 \cdot \vec{v}_2 = -2-2-2 = -6$

$\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \cdot \|\vec{v}_2\| \cos\theta \rightsquigarrow \cos\theta = \dfrac{-6}{\sqrt{3}\cdot 2\sqrt{3}} = -1$

The two lines are parallel, $\vec{v}_2 = -2\vec{v}_1$.

$\boxed{\theta = \pi \text{ rad}}$

b) (5 points) Find a parametric equation of the plane containing both lines.

$$\Pi(t,s) = (1,0,1) + t(1,1,-1) + s(-1,0,1)$$

Point $= (1,0,1)$

Tangent vectors $\left\{ \begin{array}{l} \vec{v} = (1,1,-1) \\ \vec{w} = (1,0,1) - (2,0,0) = (-1,0,1) \end{array} \right.$

c) (10 points) Find the general equation of the plane in part b), i.e. an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

for A, B, C, x_0, y_0, z_0 real numbers.

Normal vector: $\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} = (1,0,1)$

Point: $(1,0,1)$

$\boxed{1\cdot(x-1) + 0\cdot(y-0) + 1\cdot(z-1) = 0}$ or $\boxed{(x-1) + (z-1) = 0}$ or $\boxed{x + z - 2 = 0}$

d) (5 points) What is the point of intersection of this plane with the line $l_3(r) = (r, 0, 0)$?

$\left. \begin{array}{l} x + z - 2 = 0 \\ l_3(r) = (r,0,0) \end{array} \right\} \rightarrow r + 0 - 2 = 0 \Rightarrow r = 2$

$\boxed{\text{Point of intersection} = l(2) = (2,0,0)}$

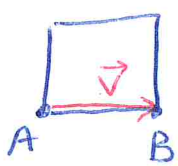
Problem 4. EXTRA CREDIT: *Doing this problem is not required. The points of this problem will be added to the total score of this exam, as long as your total score does not exceed* 100 *points.*

a) (2 points) Compute the determinant of the following matrix:

$$M = \begin{pmatrix} 2 & 1 & -1 & 0 \\ -5 & \pi & 13 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$ expand along 4^{th} column

$$\det M = 1 \cdot \det\begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = (0+0+1) - (0+0-1) = \boxed{2}$$

b) (2 points) What is the area of the square two of whose adjacent vertices are $A = (1, -1, 3)$ *and* $B = (1, 2, -1)$.



Side: $\vec{v} = B - A = (1,2,-1) - (1,-1,3) = (0,3,-4)$

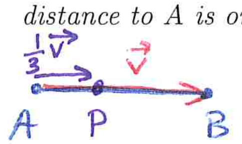
Length of side: $L = \|\vec{v}\| = \sqrt{3^2 + (-4)^2} = 5$

Area $= L^2 \cancel{\text{side}} = \boxed{25}$

c) (2 points) When computing the limit as (x, y) *goes to* $(1, 1)$ *of* $f(x, y)$ *along lines of the form* $l(t) = (1, 1) + t(a, b)$*, the result is always* 0*. Can we conclude that the limit is* 0*?*

No, this method can only be used to prove that the limit does not exist. It's possible that along some other path (not a line) we get a different limit.

d) (2 points) If $A = (1, 3, -6)$ *and* $B = (7, -3, 0)$*, what is the point* P *in the segment* AB *whose distance to* A *is one third of the distance from* A *to* B*?*



$\vec{v} = B - A = (7,-3,0) - (1,3,-6) = (6,-6,6)$

$P = A + \frac{1}{3}\vec{v} = (1,3,-6) + \frac{1}{3}(6,-6,6) = \boxed{(3,1,-4)}$

e) (2 points) What is the name of one of the most valuable prizes in mathematics (equivalent to a Nobel prize), that was awarded during our first week of classes?

The Fields Medal.