Fake Qual 4
PDE Qual Prep Seminar
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Instructions: Work 2 out of 3 problems in each of the 3 parts for a total of 6 problems.

Part 1

Problem 1. *Let* $u(x,t) = v\left(\frac{x^2}{t}\right)$.

a) *Show that*

$$u_t - u_{xx} = 0$$

if and only if

$$4zv''(z) + (2+z)v'(z) = 0, \ z > 0.$$

b) *Use part (a) to obtain the fundamental solution of the heat equation in dimension* $n = 1$.

Problem 2. *(Finite speed of propagation for solutions of the wave equation) Suppose* $u \in \mathcal{C}^2(\mathbb{R}^n \times (0,\infty))$ *solves the wave equation. Prove that if* $u = u_t = 0$ *on* $B(x_0,t_0) \times \{0\}$, *then* $u \equiv 0$ *within the cone*

$$C = \{(x,t) \colon 0 \le t \le t_0, \ |x - x_0| \le t_0 - t\}$$

Hint: *consider the energy functional*

$$e(t) = \frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2(x,t) + |Du(x,t)|^2 dx \ (0 \le t \le t_0)$$

Problem 3. *(Harnack's inequality) Show that for each connected open set* $V \subset\subset U$, *there exists a positive contant* C, *depending only on* V, *such that*

$$\sup_V u \le C \inf_V u$$

for all nonnegative harmonic functions u *in* U.

Part 2

Problem 4. *Let* $\mathbf{u}(t,x)$ *be a time-varying vector field on* $\mathbb{R}^d$*.*

a) *Define what it means for* $X : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ *to be a flow map for* $\mathbf{u}$*.*

b) *Assume that* $\mathbf{u}_1$*,* $\mathbf{u}_2$ *are Lipschitz continuous vector fields with Lipschitz constants* M_1*,* M_2 *for all* $t \in \mathbb{R}$*; that is* $|\mathbf{u}_j(t,x) - \mathbf{u}_j(t,y)| \le M_j|x-y|$*,* $j = 1,2$*, for all* $t \in \mathbb{R}$*,* $x, y \in \mathbb{R}^d$*. Let* X *be the flow map for* $\mathbf{u} := \mathbf{u}_1 + \mathbf{u}_2$ *and* X_j *be the flow map for* $j = 1,2$ *(you do not need to prove that such flow maps exist and are unique). Show that*

$$|X(t,x) - X_1(t,x)| \le M_2 t e^{M_1 t}.$$

Problem 5. *Fix* $d \ge 1$*, let* U *be an open subset of* $\mathbb{R}^d$*, and let* $p \in [1,\infty]$*. For any* $u \in W^{1,p}(\mathbb{R}^d)$ *let* $R(u) = u_{|_U}$*. Show that this defines a continuous linear operator from* $W^{1,p}(\mathbb{R}^d)$ *to* $W^{1,p}(U)$*.*

Problem 6. *Let* $\mathbf{u}(t,x)$ *be a vector field as in problem 4 in part b), and suppose that* $\rho = \rho(t,x)$ *solves*

$$\begin{cases} \partial_t \rho + \mathbf{u}\cdot\nabla\rho = \rho^2, \\ \rho(0) = \rho_0, \end{cases}$$

where ρ_0 *is continuous and bounded on* $\mathbb{R}^d$*. Suppose that* $\rho_0(x) \le M$ *for all* $x \in \mathbb{R}^d$*. Express the maximum possible time of existence of* ρ *in terms of* M*. (You need not actually prove existence, however; and you can use existence and uniqueness of flow maps for* $\mathbf{u}$ *without proof).*

Part 3

Problem 7. *Let U be a bounded domain in $\mathbb{R}^d$ with a $\mathcal{C}^\infty$ boundary. Assume that $u \in \mathcal{C}^2(\overline{U}) \cap H_0^1(U)$ is a strong solution to*

$$\begin{cases} \Delta u = u^3 + u & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Note that $u \equiv 0$ is clearly a solution, but this is a nonlinear problem, so we have no general uniqueness theorem that covers it.

a) *Use the weak maximum principle to show that $u \equiv 0$ is the only solution.*
b) *Show the same thing using an energy method.*

Problem 8. *We say that the uniformly elliptic operator*

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu,$$

satisfies the weak maximum principle if for all $u \in \mathcal{C}^2(U) \cap \mathcal{C}(\overline{U})$,

$$\begin{cases} Lu \le 0 & \text{in } U, \\ u \le 0 & \text{on } \partial U, \end{cases}$$

implies that $u \le 0$ on U.

Suppose that there exists a function $v \in \mathcal{C}^2(U) \cap \mathcal{C}(\overline{U})$ such that $Lv \ge 0$ in U and $v > 0$ on $\overline{U}$. Show that L satisfies the weak maximum principle.

Hint: *Find an elliptic operator M such that $w = y/v$ satisfies $Mw \le 0$ in the region $\{u > 0\}$. To do this, first compute $(v^2 w_{x_i})_{x_j}$.*

Problem 9. *A function $u \in H_0^2(U)$ is a weak solution of this boundary-value problem for the biharmonic equation*

$$\begin{cases} \Delta^2 u = f & \text{in } U \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

provided

$$\int_U \Delta u \Delta v dx = \int_U f v dx$$

for all $v \in H_0^2(U)$. Given $f \in L^2(U)$, prove that there exists a unique weak solution of this problem.