Fake Qual 3
PDE Qual Prep Seminar
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Instructions: Work 2 out of 3 problems in each of the 3 parts for a total of 6 problems.

Part 1

Problem 1. *Write down an explicit formula for a function u solving the equation*

$$\begin{cases} u_t - b\cdot\nabla u + cu = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases}$$

where $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constant.

Problem 2. *Let $u \in \mathcal{C}_1^2(\Omega_T)$ solve the heat equation. Prove that*

$$u(x,t) = \frac{1}{4r^n}\int\int_{E(x,t;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}dyds$$

for every heat ball $E(x,t;r) \in \Omega_T$. Note that

$$E(x,t;r) = \{(y,s)\in\mathbb{R}^{n+1}\colon s\le t,\ \frac{1}{(4\pi(t-s))^{n/2}}e^{-\frac{|x-y|^2}{4(t-s)}} \ge \frac{1}{r^n}\}$$

and

$$\frac{1}{r^n}\int\int_{E(r)}\frac{|y|^2}{s^2}dyds = 4,$$

where $E(r) = E(0,0;r)$.

Problem 3. a) *Show that the general solution of the PDE $u_{xy} = 0$ is*

$$u(x,y) = F(x) + G(y)$$

for arbitrary functions F, G.

b) *Using the change of variables $\xi = x+t$, $\eta = x-t$, show that $u_{tt} - u_{xx} = 0$ iff $u_{\xi\eta} = 0$.*

c) *Use (a) and (b) to rederive d'Alembert's formula.*

d) *Under what conditions on the initial data g, h is the solution u a right-moving wave? A left-moving wave?*

Part 2

Problem 4. *Let* $\mathbf{v} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ *be a time-varying vector field. Assume that for some* $M_1 > 0$, $\|\mathbf{v}\|_{L^\infty} \leq M_1$ *for all* $t \in \mathbb{R}$ *and for some* $M_2 > 0$, $\mathbf{v}(t)$ *has a Lipschitz constant no larger than* M_2 *for all* $t \in \mathbb{R}$.

a) *Show that for any* $(t_0, \mathbf{x}_0) \in \mathbb{R} \times \mathbb{R}^d$, *solutions to*

$$\begin{cases} \mathbf{x}'(t) = \mathbf{v}(t, \mathbf{x}(t)), \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases}$$

are unique. (You do not need to prove existence.)

b) *Define* $\mathbf{Y} : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ *by*

$$\mathbf{Y}(t_0, \mathbf{x}_0, t) = \mathbf{x}(t),$$

where $\mathbf{x}$ *is the solution from (a). Prove that* $\mathbf{Y}$ *is continuous.*

Problem 5. *Let* $U \subseteq \mathbb{R}^d$ *be open.*

a) *Integrate by parts to prove the interpolation inequality,*

$$\|\nabla u\|_{L^2} \leq \|u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}}$$

for all $u \in \mathcal{C}_c^\infty(U)$.

b) *Assume now that* U *is bounded with smooth boundary. Prove that the same inequality as in (a) holds for all* $u \in H^2(U) \cap H_0^1(U)$.

Problem 6. *Let* u *be a classical solution of the following initial boundary value problem:*

$$\begin{cases} u_t = u_{xx} & \text{in } (a,b) \times (0,T) \\ u(a,t) = u(b,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

a) *Show that the solutions are unique.*

b) *Show that there exists a constant* $\alpha > 0$ *such that*

$$\|u(\cdot,t)\|_{L^2}^2 \leq e^{-\alpha t} \|u_0\|_{L^2}^2$$

Part 3

Problem 7. *Assume u is a smooth solution of*

$$\begin{cases} u_t - \Delta u + cu = 0 & \text{in } U \times (0, \infty) \\ u = 0 & \text{on } \partial U \times [0, \infty) \\ u = g & \text{on } U \times \{t = 0\} \end{cases}$$

and suppose that $g \geq 0$ and c is bounded (but not necessarily nonnegative). Show that $u \geq 0$.

Problem 8. *Assume U is connected. Show that the only smooth solutions to the problem*

$$\begin{cases} -\Delta u = 0 & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

are $u \equiv C$ for some constant C using:

a) *energy methods.*
b) *maximum principles.*

Problem 9. *Show that a solution to the problem*

$$\begin{cases} \Delta u = u^5 - 2u^3 + 3u & \text{in } U \\ u = 1 & \text{on } \partial U \end{cases}$$

will satisfy $-1 \leq u \leq 1$.