Fake Qual 2
PDE Qual Prep Seminar
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Instructions: Work 2 out of 3 problems in each of the 3 parts for a total of 6 problems.

Part 1

Problem 1. *(Equipartition of energy) Let u solve the initial-value problem for the wave equation in one dimension:*

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}\times(0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}\times\{t=0\}. \end{cases}$$

Suppose g, h have compact support. The kinetic energy is $k(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx$ and the potential energy is $p(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx$. Prove:

a) *$k(t) + p(t)$ is constant in t,*
b) *$k(t) = p(t)$ for all large enough times t.*

Problem 2. *Assume $g \in \mathcal{C}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and let*

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)dy, (x \in \mathbb{R}^n, t > 0).$$

Prove that

$$\lim_{(x,t)\to(x_0,0^+)} u(x,t) = g(x_0)$$

for every $x_0 \in \mathbb{R}^n$.

Problem 3. *(Mean-value formulas for Laplace's equation) Prove that if $u \in \mathcal{C}^2(U)$ is harmonic, then for each ball $B(x,r) \subseteq U$ we have:*

$$u(x) = \fint_{\partial B(x,r)} u dS$$

Part 2

Problem 4. *Let* $\mathbf{F} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous in the first variable and uniformly Lipschitz continuous in the second variable with Lipschitz constant* $\gamma > 0$*, i.e. it satisfies:*

$$|\mathbf{F}(t, \mathbf{y}_1) - \mathbf{F}(t, \mathbf{y}_2)| \leq \gamma |\mathbf{y}_1 - \mathbf{y}_2|$$

for any $\mathbf{y}_i \in \mathbb{R}^n$ *and for any* $t \in \mathbb{R}$*. Let* $\mathbf{y}_1$ *and* $\mathbf{y}_2$ *be solutions in the interval* $[t_0, T]$ *to the problem:*

$$\mathbf{y}_i'(t) = \mathbf{F}(t, \mathbf{y}_i(t))$$

for $i = 1, 2$ *and* $t \in [t_0, T]$*.*

a) *Show that:*

$$|\mathbf{y}_1(t) - \mathbf{y}_2(t)|^2 \leq |\mathbf{y}_1(t_0) - \mathbf{y}_2(t_0)|^2 e^{2\gamma(t-t_0)}.$$

b) *Use this to prove uniqueness of solution of the corresponding Initial Value Problem.*

Problem 5. a) *Using the separation of variables method, find a (formal) solution of the periodic heat problem:*

$$\begin{cases} u_t - k u_{xx} = 0 & 0 < x < 2\pi, \ t > 0, \\ u(0,t) = u(2\pi, t), \ u_x(0,t) = u_x(2\pi, t) & t \geq 0, \\ u(x, 0) = f(x) & 0 \leq x \leq 2\pi, \end{cases}$$

where f *is a smooth periodic function.*

b) *Show that if* v *is an arbitrary partial derivative of the solution* u*, then* $v(0,t) = v(2\pi, t)$*, for all* $t \geq 0$*.*

Problem 6. *Let* U *be the open unit ball in* $\mathbb{R}^d$*.*

a) *Let*

$$u(x) = |x|^{-\alpha}.$$

For which values of $\alpha > 0$*,* $d \geq 1$*, and* $p > 1$ *does* u *belong to* $W^{1,p}(U)$*?*

b) *Show that*

$$u(x) = \log\log\left(1 + |x|^{-1}\right)$$

belongs to $W^{1,2}(U)$ *but does not belong to* $L^\infty(U)$*.*

Part 3

Problem 7. *Prove that there is at most one smooth solution of*

$$\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \times [0, T] \\ u = g & \text{on } U \times \{0\} \end{cases}$$

Problem 8. *Let u be a solution of $\Delta u = u^3 - u$ on a bounded domain Ω. Assume that $u = 0$ on $\partial\Omega$. Show that $u \in [-1, 1]$ throughout Ω. Can the values ± 1 be achieved?*

Problem 9. *Consider the Poisson equation with Dirichlet boundary condition:*

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

where U is a bounded, open subset of $\mathbb{R}^n$ and $f \in L^2(U)$. We know there exists a weak solution $u \in H_0^1(U)$. Prove that $u \in H_{loc}^2(U)$.