Fake Qual 1
PDE Qual Prep Seminar
Xavier Ramos Olivé
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Instructions: Work 2 out of 3 problems in each of the 3 parts for a total of 6 problems.

Part 1

Problem 1. *Give a direct proof that if U is bounded and $u \in \mathcal{C}_1^2(U_T) \cap \mathcal{C}(\bar{U_T})$ solves the heat equation, then*

$$\max_{\bar{U_T}} u = \max_{\Gamma_T} u.$$

Hint: *Define $u_\epsilon := u - \epsilon t$ for $\epsilon > 0$, and show u_ϵ cannot attain its maximum over $\bar{U_T}$ at a point in U_T.*

Problem 2. *Let $u_1, u_2 \in \mathcal{C}^2(\Omega)$ solve the Laplace equation*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases}$$

Prove that $u_2 - u_1 = c$, where c is constant.

Problem 3. *(Wave equation in the half line) Recall that d'Alembert's formula:*

$$v(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy$$

provides us with a solution to the problem:

$$\begin{cases} v_{tt} - v_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ v = g, \ v_t = h & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

Consider the wave equation in the half line:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = g, \ u_t = h & \text{on } \mathbb{R}_+ \times \{0\} \\ u = 0 & \text{on } \{0\} \times (0,\infty), \end{cases}$$

where g, h are given, with $g(0) = h(0) = 0$. Let $\tilde{u}, \tilde{g}, \tilde{f}$ be the odd extensions of u, g, f to $\mathbb{R}$.

a) *Show that $\tilde{u}$ solves:*

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ \tilde{u} = \tilde{g}, \ \tilde{u}_t = \tilde{h} & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

and solve for $\tilde{u}$.

b) *Find $u(x,t)$.*

Part 2

Problem 4. *Let $A(t)$ be a continuous function from t in $\mathbb{R}$ to the space of square, real-valued matrices.*

a) *Show that for every solution of the (non-autonomous) linear system, $\dot{\mathbf{x}} = A(t)\mathbf{x}$, we have*
$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}(0)\| e^{\int_0^t \|A(s)\| ds},$$
where $\|A(s)\|$ is the operator norm and $\|\mathbf{x}(t)\|$ is the usual Euclidean norm.

b) *Show that if $\int_0^t \|A(s)\| ds < \infty$ then every solution, $\mathbf{x}(t)$, has a finite limit as $t \to \infty$.*

Problem 5. *Solve the following wave equation using the Fourier transform:*
$$\begin{cases} \partial_{tt} u - \partial_{xx} u = 0 & \text{for } t > 0, x \in \mathbb{R}, \\ u(0,x) = \frac{1}{1+x^2}, \ \partial_t u(0,x) = 0. \end{cases}$$
(The solution will be in the form of an integral that you need not try to integrate in closed form.)

Problem 6. *Let $u(x,y)$ be a solution to the equation:*
$$\mathbf{b}(x,y) \cdot \nabla u(x,y) + u = 0$$
for $\mathbf{b} = (a(x,y), b(x,y))$ a vector field whose components are positive and differentiable functions in $\mathbb{R}^2$. Define:
$$D = \{(x,y) \in \mathbb{R}^2 \colon |x| < 1, |y| < 1\}.$$

a) *Prove that the projection on the $x,y-$plane of each characteristic curve passing through a point in D intersects the boundary of D at exactly two points.*

b) *Show that if u is positive on the boundary of D, then it is positive at every point in D.*

c) *Suppose that u attains a local minimum (maximum) at a point $(x_0, y_0) \in D$. Evaluate $u(x_0, y_0)$.*

d) *Denote by m the minimal value of u on the boundary of D. Assume $m > 0$. Show that $u(x,y) \geq m$ for all $(x,y) \in D$.*

Remark: *This is an atypical example of a first-order PDE for which a maximum principle holds true.*

Part 3

Problem 7. *Suppose u is a smooth solution to*

$$\begin{cases} \Delta u = u^3 + u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

Show that $u \equiv 0$ in U using maximum principle.

Problem 8. *Suppose u is a smooth solution to*

$$\begin{cases} u_t - \Delta u + cu = 0 \\ u = 0 \\ u = g \end{cases}$$

where the function c satisfies $c \geq \delta > 0$. Show that $|u(x,t)| \leq Ce^{-\gamma t}$.

Problem 9. *Assume U is connected. Consider the Poisson's equation with the Neumann boundary-value problem*

$$\begin{cases} -\Delta u = f, & \text{on } U, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial U. \end{cases}$$

a) *State the definition of the weak solution to this problem;*
b) *Show that for $f \in L^2(U)$, the above problem has a weak solution if and only if $\int_U f dx = 0$;*
c) *If $f = 0$, use the energy method to show that the only smooth solutions of this problems are $u = C$, for some constant C.*