Extra Credit Homework
MATH 10A, Summer Session E, UC Riverside 2018

Problem 1. *Euclidean space of dimension 4, $\mathbb{R}^4$, can be described as the set of points labeled with quadruples (x, y, z, w), where x, y, z, w are real numbers. In analogy to what happens in $\mathbb{R}^2$ and $\mathbb{R}^3$, we can think of vectors $\mathbf{v} = (v_1, v_2, v_3, v_4)$ as arrows that bring us from one point to another. The basic operations with vectors in $\mathbb{R}^4$ can be summarized as:*

(1) $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3, v_4 + w_4)$
(2) $\alpha\mathbf{v} = (\alpha v_1, \alpha v_2, \alpha v_3, \alpha v_4)$
(3) $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 + v_4 w_4$
(4) $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2}$

where $\mathbf{v} = (v_1, v_2, v_3, v_4)$ and $\mathbf{w} = (w_1, w_2, w_3, w_4)$ are any two vectors in $\mathbb{R}^4$, and α is any real number.

a) *Knowing that the formula $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$ still holds (as the proof we did in class works in any dimension), find the angle between the vectors $\mathbf{v} = (1, 0, -1, 2)$ and $\mathbf{w} = (0, 1, 2, 1)$.*

b) *Show that the vector $\mathbf{u} = (1, -2, 1, 0)$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.*

c) *Consider the parametric equations of the plane $\pi(t, s) = (1+t, 2+s, -1-t+2s, 2t+s)$ and the line $l(r) = (3+r, -1-2r, r, 2)$. The direction vector of the line is $\mathbf{u}$ and the direction vectors of the plane are $\mathbf{v}$ and $\mathbf{w}$, hence the line is perpendicular to the plane (as you showed in part (b)). Show that, despite this, the line does not intersect the plane.*

Remark: *in $\mathbb{R}^3$, a line and a plane are either parallel, or they intersect at a point. In particular, if a line is perpendicular to a plane, it will always intersect it. In $\mathbb{R}^4$, however, there is enough room for a plane and a line to cross each other, without intersecting, and without being parallel (in a similar way that two lines in $\mathbb{R}^3$ might cross each other, although their direction vectors might be perpendicular).*

Problem 2. *Given a function $f:\mathbb{R}^2\to\mathbb{R}$, we define the Laplacian of f to be*

$$\Delta f := \frac{\partial^2 f}{\partial^2 x}+\frac{\partial^2 f}{\partial^2 y}$$

This is a very common differential operator, that appears in many areas of physics: to describe waves, to study the electric or the gravitational potential, to talk about kinetic energy in quantum mechanics, etc.

Sometimes, when studying a physical problem, one assumes that the problem has some kind of symmetry; for instance, we could assume that a given problem has rotational symmetry. If that was the case, polar coordinates would be a better way of describing the problem than cartesian coordinates. However, if we write our function in polar coordinates $f=f(r(x,y),\theta(x,y))$, computing the Laplacian of f is not the same as taking second derivatives with respect to r and with respect to θ and adding them together, that is:

$$\Delta f(r,\theta)\neq \frac{\partial^2 f}{\partial^2 r}+\frac{\partial^2 f}{\partial^2 \theta}$$

a) *Consider the function $f(x,y)=x^2+y^2$. Compute Δf and write the resulting function in polar coordinates. Then, write f in polar coordinates, and compute $\frac{\partial^2 f}{\partial^2 r}+\frac{\partial^2 f}{\partial^2 \theta}$. Show that the results are not the same (this is an example of the assertion above).*

b) *Knowing that $r=\sqrt{x^2+y^2}$, show that:*

$$\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r}$$
$$\frac{\partial^2 r}{\partial x^2}=\frac{1}{r}-\frac{x^2}{r^3},\ \frac{\partial^2 r}{\partial y^2}=\frac{1}{r}-\frac{y^2}{r^3}$$

c) *Using that $x=r(x,y)\cos(\theta(x,y))$, take derivatives in both sides with respect to x to show:*

$$\frac{\partial \theta}{\partial x}=-\frac{\sin\theta}{r},\ \frac{\partial^2\theta}{\partial^2 x}=\frac{2\sin\theta\cos\theta}{r^2}$$

d) *Do the same as in part c), this time taking derivatives with respect ot y in $y=r(x,y)\sin(\theta(x,y))$, to show:*

$$\frac{\partial \theta}{\partial y}=\frac{\cos\theta}{r},\ \frac{\partial^2\theta}{\partial^2 y}=-\frac{2\sin\theta\cos\theta}{r^2}$$

e) *Use chain rule twice in $f(r(x,y),\theta(x,y))$ to show:*

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 f}{\partial^2 r}\left(\frac{\partial r}{\partial x}\right)^2+\frac{\partial f}{\partial r}\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 f}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2+\frac{\partial f}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}+2\frac{\partial^2 f}{\partial r\partial\theta}\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x}$$

and

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 f}{\partial^2 r}\left(\frac{\partial r}{\partial y}\right)^2+\frac{\partial f}{\partial r}\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 f}{\partial \theta^2}\left(\frac{\partial \theta}{\partial y}\right)^2+\frac{\partial f}{\partial \theta}\frac{\partial^2 \theta}{\partial y^2}+2\frac{\partial^2 f}{\partial r\partial\theta}\frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y}$$

f) *Use the results above, to conclude that:*

$$\Delta f(r,\theta)=\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

g) *Finally, use the formula in f) to compute Δf of the function in a). I.e., write the function $f(x,y)=x^2+y^2$ in polar coordinates, and use the formula above to compute Δf.*