2/4/2016

MATH CLUB | Discovering a Theorem by Erdös

Today I am going to talk you about something that happened to me last quarter. I was relaxing at home, checking facebook, when a friend of mine cited (retweeted) @AlgebraFact a result by Erdös that said:

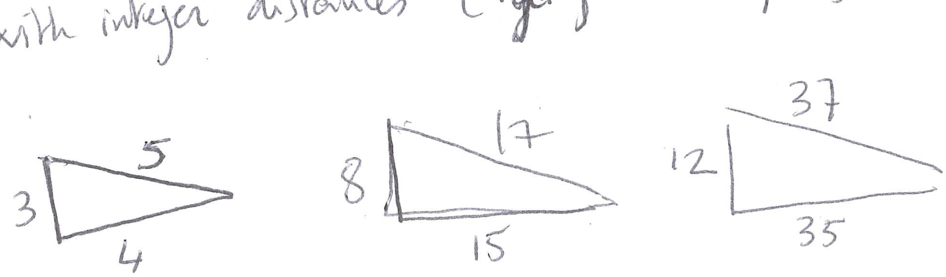
"If an infinite set of points in the plane determines only integer distances, then all the points lie on a straight line." ← Erdös - Anning Thm (1945)

After realizing that this is a nice, surprising, elegant, impressive result, the first question that came to my mind was: is it possible if we have finitely many points?

↳ Is it possible to find n points in the plane which do not lie in a line whose distances are all integer numbers?

Well, for $n=3$ it's easy, we know there exist right triangles with integer distances (Pythagorean triples):

Infinitely many



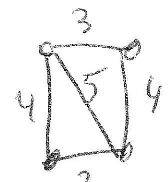
But is it possible to do it for an arbitrary number n?

The answer is YES, and you can see a proof in Erdős-Anning paper (1945) where they construct n points on a circle defining rational distances (using some relations they know about primes numbers), and expanding then the circle (homothety multiplying all distances by common denom.) to obtain integer dist.

But I didn't know the answer, so I spend my afternoon figuring out a proof.

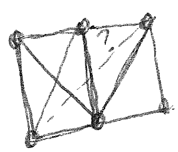
$\boxed{n=4}$

Use two triangles with integer sides:



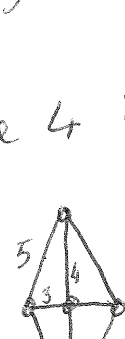
$\boxed{n=5}$ → We have to be more careful as more and more distances have to be taken into account

Example



← Things go wrong easily.

Idea: use 4 Pythagorean triangles to create a rhombus:

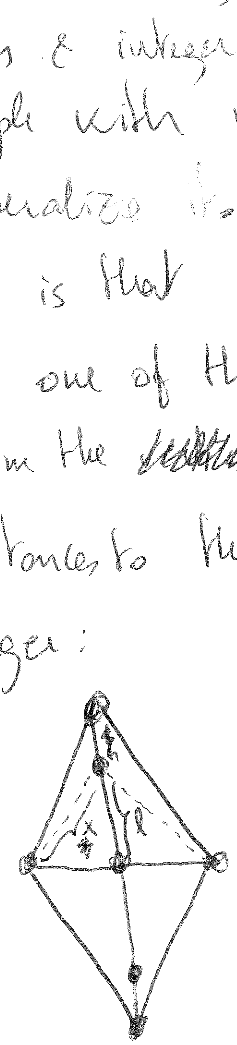


← example with n=5.

At this point I started to ~~believe~~ have some hope that maybe it was possible to build examples for arbitrary n. I wanted to construct an example for larger n's; ~~for~~ to do so, I needed to exploit ~~[illegible]~~ constructions with a lot of symmetry to be able to keep track of all the distances (a simple combinatorics computation gives you $\binom{n}{2} = \frac{n(n-1)}{2}$ distances to check for an example with n points).

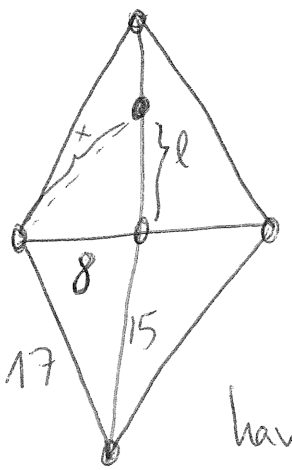
I thought about using circles (for example, a regular ~~hex~~ hexagon inscribed in a circle to get an example with $n=7$); but this didn't work (it's actually impossible to get hexagon with ~~a~~ integer radius & integer distances). Hence I looked at the example with $n=5$, and I realized that I could try to generalize it.

The idea is that we can try to add points in ~~the long~~ one of the two diagonals at integer distances from the ~~vertex~~ origin, in such a way that the ~~two~~ distances to the 2 vertexs in the other diagonal are integer:



If we do so, we only have to keep track of 2 distances (or, really, just 1).

In the construction above, this can not be done because the original triangles are too small (the smallest side of the triangle, 3, doesn't have enough divisors). But let's see what happens with the Pythagorean triple $(8, 15, 17)$.



$$x^2 = l^2 + 8^2$$
$$\Downarrow$$
$$8^2 = x^2 - l^2 = (x+l)(x-l)$$

Since $x, l \in \mathbb{Z}$, $x+l, x-l \in \mathbb{Z}$, so we have to consider all the possible solutions where $x+l, x-l$ are complementary factors of $8^2 = 64$.

Since $x+l > x-l$:

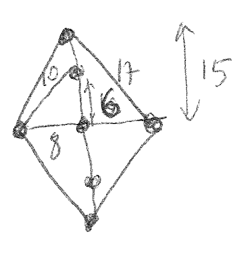
$$\begin{cases} x+l = 64 \\ x-l = 1 \end{cases} \Downarrow \quad 2x = 65 \quad \times$$
$\downarrow$ gives a non-integer solution.

$$\begin{cases} x+l = 32 \\ x-l = 2 \end{cases} \Downarrow \quad \begin{aligned} 2x = 34 &\Rightarrow x = 17 \\ 2l = 30 &\Rightarrow l = 15 \quad \times \end{aligned}$$
$\downarrow$ Corresponds to an existing point

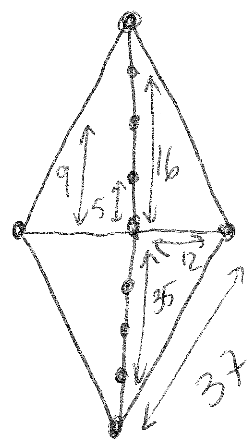
$$\begin{cases} x+l = 16 \\ x-l = 4 \end{cases} \Downarrow \quad \begin{aligned} 2x = 20 &\Rightarrow \boxed{x = 10} \\ 2l = 12 &\Rightarrow \boxed{l = 6} \end{aligned}$$
NICE!

$$\begin{cases} x+l = 8 \\ x-l = 8 \end{cases} \Downarrow \quad \begin{aligned} 2x &= 16 \\ 2l &= 0 \quad \times \end{aligned}$$

This allows a construction with $n = 7$.



You can do ~~the~~ the same construction with a rhombus build out of 4 triangles of sides 12, 35 & 37. You will see that in that case it is possible to get an example with n = 11 points.



At this point, I started to believe that it is possible to construct an example with n points for an arbitrary n. However, I needed a way to quantify the number of points that could be added, and so to ~~b~~ keep ~~...~~ under control the possible divisors of the length of the ~~...~~ side where we are not adding points.

One way to keep it under control is to look for triangles with length a power of 2, say 2^k.

Question: Is ~~...~~ it always possible to find Pythagorean triples with one cathetus of side 2^k?

The answer to that is YES and it follows from Euclid's formula. ~~However~~ Honestly, I had no idea about that, so I looked at Wikipedia, and I found this magical formula, that says that given two positive integers m, n, with $m > n$, then:

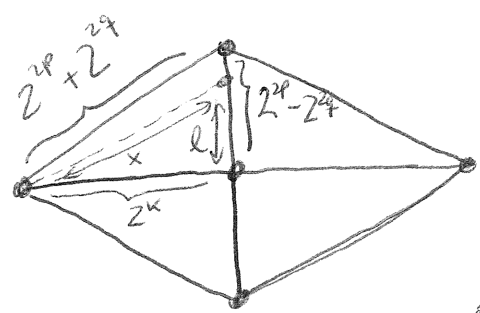
$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2$$

form a Pythagorean triple.

↳ Check that easily, if you don't belive in Wikipedia: $a^2 + b^2 = c^2$.

Then we can easily make $b = 2^K$, by choosing $m = 2^p$, $n = 2^q$, in such a way that $p + q + 1 = K$, $p > q$.

Then we have:



$$x^2 = \ell^2 + 2^{2K} \longrightarrow x^2 - \ell^2 = 2^{2K} \longrightarrow (x+\ell)(x-\ell) = 2^{2K}$$

Hence, we have to solve the systems:

$$\begin{cases} x + \ell = 2^{2K-i} \\ x - \ell = 2^i \end{cases} \quad \text{for } i = 0, 1, \dots, K$$

(as we can always assume $x + \ell \geq x - \ell$).

Now you see what I mean by "Keeping track of the divisors"; if we don't do that, we have to consider all possible combinations of divisors.

We can solve explicitly the system:

$$\begin{cases} x+\ell = 2^{2K-i} \\ x-\ell = 2^{i} \end{cases} \Rightarrow \begin{cases} 2x = 2^{2K-i} + 2^{i} \\ 2\ell = 2^{2K-i} - 2^{i} \end{cases} \Rightarrow \begin{cases} x = 2^{2K-i-1} + 2^{i-1} \\ \ell = 2^{2K-i-1} - 2^{i-1} \end{cases}$$

Now we see that there are some undesired solutions, as in the example we saw before; but actually, they are the same as before!

Non-integer solutions:

~~[illegible]~~

[illegible]

$i-1 = 0-1 \Rightarrow \boxed{i=0}$

$\ell = 0$

$2K-i-1 = i-1$

$2K = 2i$

$\boxed{i=K}$

Point in the center

$2K-i-1 = 2p$

$2(p+q+1)-i-1 = 2p$

$\boxed{2q+1 = i}$

$\Downarrow$

$\ell = 2^{2p} - 2^{2q}$

Point on the top

Hence we only get $K+1-3 = K-2$ possible solutions, which allow us to add $2\cdot(K-2)$ points to the existing 5 points, giving a construction with $n = 2K+1$ points for any K.

↳ This is true assuming that all the solutions are different; to see that, notice that:

$$2^{2K-i-1} - 2^{i-1} = 2^{2K-j-1} - 2^{j-1}$$

$$\Updownarrow \cdot 2^{-K+1}$$

$$2^{K-i} - 2^{i-K} = 2^{K-j} - 2^{j-K}$$

$$\Updownarrow$$

$$e^{\ln 2 (K-i)} - e^{-\ln 2 (K-i)} = e^{\ln 2 (K-j)} - e^{-\ln 2 (K-j)}$$

$$\Updownarrow$$

$$\sinh(\ln(2)\cdot(K-i)) = \sinh(\ln(2)(K-j)) \iff i = j$$

sinh(x) is an injective function.

This completes the proof, because it's giving a construction with any odd number of points (and then, removing one arbitrary point, we get a construction with an arbitrary even number of points).